Semi-divisible rings

Chancelle O. Kamga   Celestin Lele
Jean B. Nganou   Lutz Strüngmann

Presenter: Jens Kötters

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Definition of a bounded residuated lattice

A *bounded commutative integral residuated lattice*, (shortly *residuated lattice*) is a nonempty set L equipped with four binary operations $\wedge, \vee, \odot, \longrightarrow$ and two constants $0, 1$ satisfying the following axioms:

(C1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $(\odot, \longrightarrow)$ is an adjoint pair, that is, $x \odot y \leq z$ iff $x \leq y \rightarrow z$, for all $x, y, z \in L$ (residuation).

Rings as natural examples

Given any commutative ring with unity R, the lattice of ideals of R forms a residuated lattice

$$A(R) := \langle \mathrm{Id}(R), \wedge, \vee, \odot, \rightarrow, \{0\}, R \rangle,$$

where:

$$\begin{aligned} I \wedge J &:= I \cap J, \\ I \vee J &:= I + J, \\ I \odot J &:= I \cdot J, \\ I \rightarrow J &:= \{x \in R : xI \subseteq J\}. \end{aligned}$$

Note that $I^* := \{x \in R : xI = 0\}$ is simply the annihilator of I in R.

Notable classes of residuated lattices

A residuated lattice $\mathcal{L}$ is called:

1. *Heyting algebra* if it satisfies the condition *(I)*: $x \odot x = x$, for all $x \in L$ (idempotence). Note that in a Heyting algebra the operations $\odot$ and $\wedge$ coincide;
2. *MTL algebra* if it verifies the condition *(P)*: $(x \to y) \vee (y \to x) = 1$, for all $x, y \in L$ (prelinearity);
3. *Divisible or Rℓ-monoid* if it verifies the divisibility condition *(D)*: $x \odot (x \to y) = x \wedge y$, for all $x, y \in L$ (divisibility);
4. *Regular* if it verifies the condition *(DN)*: $x^{**} = x$, for all $x \in L$ (double negation);
5. *BL-algebra* if it is and Rℓ-monoid and an MTL-algebra;
6. *MV-algebra* if it is a regular BL-algebra.

Definition of a semi-divisible residuated lattice

- A given residuated lattice $\mathcal{L}$ is said to be *semi-divisible* if it verifies for all $x, y \in L$,

$$\textit{(SD)} \qquad (x^* \odot (x^* \to y^*))^* = (x^* \wedge y^*)^*$$

- Observe that condition *(D)* implies condition *(SD)*, in other words, divisible residuated lattices are semi-divisible). But the converse is not always true [1, 5].

A characterization of semi-divisibe residuated lattices

Let $\mathcal{L}$ be a residuated lattice. The following conditions are equivalent.

1. $\mathcal{L}$ is a semi-divisible residuated lattice.

2. For every $x, y, z \in L$,
   $z^* \rightarrow (x^* \odot (x^* \rightarrow y^*))^* = z^* \rightarrow (x^* \wedge y^*)^*$

Note: Heyting algebras, MV-algebras and BL-algebras are divisible and consequently semi-divisible.

Semi-divisible and not divisible

Consider a structure $L_{SD} = ([0,1], \wedge, \vee, \odot, \rightarrow, 0, 1)$ such that, for all $x, y \in [0,1]$,

$$x \odot y = \begin{cases} 0 & x, y \in [0, \frac{1}{2}], \\ min\{x, y\} & elsewhere \end{cases}$$

$$x \rightarrow y = \begin{cases} 1 & if \;\; x \leq y, \\ \frac{1}{2} & if \;\; y < x \leq \frac{1}{2}, \\ y & if \;\; y < x, \frac{1}{2} < x. \end{cases}$$

$MV(L_{SD}) = \{0, \frac{1}{2}, 1\}$ which is an MV-algebra.
On can verify that L_{SD} is a semi-divisible residuated lattice.
However, L_{SD} is not divisible. Infact, taking $x = \frac{1}{3}, y = \frac{1}{2}$, then $\frac{1}{2} \odot (\frac{1}{2} \rightarrow \frac{1}{3}) = \frac{1}{2} \odot \frac{1}{2} = 0 \neq \frac{1}{3} = min(\frac{1}{2}, \frac{1}{3})$.

Definition and notations

Given any commutative ring with unity R, the set $\mathrm{Id}(R)$ of ideals of R carries a natural structure of a residuated lattice

$$A(R) := \langle \mathrm{Id}(R), \wedge, \vee, \odot, \rightarrow, \{0\}, R \rangle,$$

where:

$$\begin{aligned} I \wedge J &:= I \cap J, \\ I \vee J &:= I + J, \\ I \odot J &:= I \cdot J, \\ I \rightarrow J &:= \{x \in R : xI \subseteq J\}. \end{aligned}$$

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More notations

- Given a commutative ring R, an ideal I of R is called:
  - an *annihilator ideal* if $I = J^*$ for some ideal J of R;
  - a *dense ideal* if $I^* = 0$.
- $A(R)$ denotes the residuated lattice of ideals of R.
- $\mathrm{MV}(R)$ denotes the set of annihilator ideals of R.

Definition of SD-rings

A commutative ring with unity R is a semi-divisible ring (SD-ring) if $A(R)$ satisfies the following identity for all $I, J \in Id(R)$:

$$(I^* \cdot (I^* \to J^*))^* = (I^* \cap J^*)^*$$

Where $I \to J = \{x \in R : xI \subseteq J\}$ and $I^* = I \to 0$.

Key remarks

Given a commutative ring R and let $I, J \in Id(R)$;

1. if $I \subseteq J$ then $J^* \subseteq I^*$
2. $(I^* \cap J^*)^* \subseteq (I^* \cdot (I^* \to J^*))^*$

It follows that a sufficient condition for a ring R to be a semi-divisible ring is that $(I^* \cdot (I^* \to J^*))^* \subseteq (I^* \cap J^*)^*$ for all ideals I, J of R.

Note: R is an SD-ring if and only if $\mathrm{Id}(R)$ is a semi-divisible residuated lattice.

Classes of SD-rings

Recall the following previously treated classes of rings.

- A commutative ring with unity R, the set $A(R)$ is an MV-algebra if and only if R is a Lukasiewicz ring [2] (that is; a unitary ring satisfying $I + J = (I^* \cdot (I^* \cdot J)^*)^*$ for all $I, J \in Id(R)$).

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- A commutative ring R is called a *multiplication ring* [3] if, for every pair of ideals I, J of R with $I \subseteq J$, there exists an ideal K of R such that $I = J \cdot K$
- A commutative ring with unity R, the set $A(R)$ is a BL-algebra if and only if R is a BL-ring [4].
- Note: Lukasiewicz rings, BL-rings, multiplication rings are all SD-rings.

BL-rings versus SD-rings

It is clear that BL-rings are SD-rings. The following example shows that the class of SD-rings is larger.

- Let
$$R = \mathbb{F}_2[x, y]/(x^2, xy, y^2),$$
where $\mathbb{F}_2$ denotes the finite field with two elements

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- The ideals of the R are
$$\begin{aligned}
(0) &= \{0\},\\
(x) &= \{0, x\},\\
(y) &= \{0, y\},\\
(x+y) &= \{0, x+y\},\\
(x, y) &= \{0, x, y, x+y\},\\
(1) &= R \quad \text{(the unit ideal)}.
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- It can be verified that R is an SD-ring that is not a BL-ring, not even a multiplication ring.

Characterizations of SD-rings

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- $(I^* \to J^*) \to J^* = (J^* \to I^*) \to I^*$, for every $I, J \in Id(R)$.

Other properties of SD-rings

Let R be an SD-ring and $I, J \in Id(R)$, then

1. If $J^{**} \subseteq I^*$, then $I^* \to (I^* \odot J^*)^{**} = J^*$;
2. $(I \odot J)^* = (I^{**} \cap J^*)^* \to I^* = I^{**} \to (I^{**} \cap J^*)^{**}$

Related properties

Let R be a commutative ring with unity and $I \in Id(R)$. We consider the properties:

$$T_1 : (I^* \odot I^*)^* = I^{**} \quad \text{for all } I \in \mathrm{Id}(R);$$

$$T_2 : (I^* \odot J^*)^* = (I^* \cap J^*)^* \quad \text{for all } I, J \in \mathrm{Id}(R).$$

$$T_3 : (I^* \odot I^*)^* = I^{**} \quad \text{for all } I \in \mathrm{Id}(R).$$

Prop.: $T_1 \Leftrightarrow T_2$ and $T_3 \Rightarrow$ SD-ring. **Remark.** An SD-ring needs

not satisfy T_3. For instance, in the SD-ring presented earlier, the ideal (x, y) does not satisfy T_3.

SD-rings: products, direct sums and quotients

Proposition

The class of SD-rings is closed under each of the following operations:

1. *Finite direct products,*
2. *Arbitrary direct sums,*
3. *Homomorphic images.*

SD-rings versus Baer rings

Recall the definition of a *Baer ring*.

A Baer ring is a ring in which every annihilator ideal is principal and generated by an idempotent. That is, for every ideal I of R, there exists an idempotent element $e \in R$ such that

$$I^* = \mathrm{Ann}(I) = eR.$$

- In any Baer ring R, for each element $a \in R$, there exists a unique idempotent element $a^* \in R$ such that $(aR)^* = a^* R$

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- In any Baer ring R, for each element $a \in R$, there exists a unique idempotent element $a^* \in R$ such that $(aR)^* = a^*R$
- **Prop.** Every Baer ring is an SD-ring.
- **Remark.** Baer rings are not necessarily multiplication rings. Indeed, $R = \prod_{i \in I} \mathbb{F}_2$ is an example of a Baer ring that is not a multiplication ring.

SD-rings versus reduced rings

Recall that a reduced ring is a ring with no nonzero nilpotent elements.

- **Proposition.** Every unitary commutative reduced ring is an SD-ring.

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Recall that a reduced ring is a ring with no nonzero nilpotent elements.

- **Proposition.** Every unitary commutative reduced ring is an SD-ring.

- **Proof** Suppose that R is a reduced ring. Let $I \in Id(R)$. If $a \in I \cap I^*$, then $a^2 = 0$. Since a is not 2-nilpotent, we have $a = 0$. This shows that $I \cap I^* = 0$ for all $J \in Id(R)$. So we obtain

$$\begin{aligned} I^* &= I \to (I \cap I^*) \\ &= (I \to I) \cap (I \to I^*) \\ &= R \cap (I \to I^*) \\ &= I \to I^* \\ &= (I \odot I)^*. \end{aligned}$$

For all $I \in Id(R)$, we have $I^* = (I \odot I)^*$, which implies that $I^{**} = (I^* \odot I^*)^*$ and the result follows by T_3.

Summary

We have formulated the definition of an SD-ring and shown that each of the following classes of rings is contained in the class of SD-rings.

- Łukasiewicz rings

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- Łukasiewicz rings
- BL-rings
- Multiplication rings
- Baer rings
- Reduced rings
- **Note:** Each of the containments above is proper.

Yet to achieve!

We are working on finding some type of representation results for SD-rings.

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THANK YOU FOR YOUR ATTENTION !!!!!!!!