Makkai's lost proof

of projectivity of N in the free topos

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The constructivist's bargain

REJECT choice axioms

$\leftrightsquigarrow$

COLLECT definability principles

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Admissible for many constructive systems: intuitionistic higher-order logic (IHOL_N), first-/second-/higher-order Heyting arithmetic, most type theories, some set theories…

Generalisations: disjunction property, numerical existence property, class existence property…

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Today: Intuitionistic higher-order logic IHOL_N (1, $\times$, $=$, P, N) admits rule of countable choice RC_ω:

$$\frac{\vdash \forall n : N \ \exists x : X \ \varphi(n, x)}{\vdash \exists f : X^N \ \forall n : N \ \varphi(n, f(n))} \ \mathrm{RC}_\omega$$

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$$\begin{tikzcd} A \arrow[d, two heads, "e"] \\ N \arrow[u, dashed, bend left, "\exists s"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \llbracket\, n{:}N,\ x{:}A \mid \varphi(n,x)\,\rrbracket \arrow[d, two heads, "\pi_1"] \\ N \arrow[u, dashed, bend left, "\exists f"] \end{tikzcd}$$

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in the free topos $\mathcal{F}$, categorical avatar of IHOL_N.

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- 2026 Forssell, Lumsdaine and Swan 2026, *'Makkai's lost proof of projectivity of N in the free topos'*, arXiv:2604.01139

EP for IHOL: Freyd's classic gluing proof

("Topos" = "elementary topos with NNO" throughout.)

Proposition

The terminal object of the free topos $\mathcal{F}$ is projective: IHOL_N *admits* EP.

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Proof.

For any left-exact functor of toposes $F : \mathcal{C} \longrightarrow \mathcal{D}$, the comma category $(\mathcal{D} \downarrow F)$ is a topos, and its projection to $\mathcal{C}$ is logical. Apply to "global sections" $\Gamma : \mathcal{F} \longrightarrow \mathrm{Set}$:

$$\begin{tikzcd}
(\mathrm{Set} \downarrow \Gamma) \arrow[r, "(-)_1"] \arrow[d, "(-)_0"] & \mathrm{Set} \\
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- Suppose IHOL_N proves '$\forall x{:}N\ \exists y{:}A\,\varphi(x,y)$'.
- Compactness: proof uses finitely many iterations of P, lies in some finite-rank fragment $\mathrm{IHOL}^r_N \subseteq \mathrm{IHOL}_N$.
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Outline proof, categorialised

- Suppose $e : A \twoheadrightarrow N$ cover in free topos $\mathcal{F}$.
- Compactness: for some $r \in \mathbb{N}$, e lifts to the free *r-ranked topos* $\mathcal{F}_r \subseteq \mathcal{F}$.
- Free structures internalise: $\mathcal{F}$ sees a cover $\ulcorner e \urcorner : \ulcorner A \urcorner \twoheadrightarrow \ulcorner N \urcorner$ in the internal free r-topos $\mathcal{F}_r^{\mathcal{F}}$.
- Specialise to numerals: $\mathcal{F}/[n{:}N]$ has $\ulcorner e \urcorner_{\bar{n}} : \ulcorner A \urcorner_{\bar{n}} \twoheadrightarrow 1$ in its free r-topos $\mathcal{F}_r^{\mathcal{F}/N}$.
- $\mathcal{F}/N$ has standard model of $\mathcal{F}_r^{\mathcal{F}}$, so internalises Freyd gluing proof of EP for $\mathcal{F}_r$, and so believes 'there exists a section of $\ulcorner e \urcorner_{\bar{n}}$ in $\mathcal{F}_r^{\mathcal{F}/N}$'.
- $\mathcal{F}/N$ can see $\mathcal{F}_r^{\mathcal{F}/N}$ as Gödel-numbered, so this existence is witnessed by $1 \longrightarrow \mathrm{mor}\, \mathcal{F}_r^{\mathcal{F}/\mathbb{N}}$ in $\mathcal{F}/N$, i.e. $a : N \longrightarrow \mathrm{mor}\, \mathcal{F}_r^{\mathcal{F}}$ in $\mathcal{F}$.
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Ranked toposes

Definition

An *r-ranked topos* ($r \in \mathbb{N} \cup \{\infty\}$): a regular category $\mathcal{E}$ with NNO, with full subcats $\mathcal{E}_0 \subseteq \cdots \mathcal{E}_i \cdots \subseteq \mathcal{E}$ ($0 \leq i < r$) such that:

1. each $\mathcal{E}_i$ is regular subcat with NNO;
2. for $0 \leq i < r - 1$, each object of $\mathcal{E}_i$ has a power-object in $\mathcal{E}$, in $\mathcal{E}_{i+1}$.

Idea: like toposes, but with only r iterations of power-objects.

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Proposition

Free r-toposes $\mathcal{F}_r$ are just as nice as the free topos $\mathcal{F}$:

- *strictly initial in an essentially-algebraic (1-)category of r-toposes and strict logical functors;*
- *bi-initial in 2-category of r-toposes and logical functors.* □

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Proof.

Go via free ∞-ranked topos $\mathcal{F}_\infty$:

1. $\varinjlim_r \mathcal{F}_r \simeq \mathcal{F}_\infty$ by compactness for essentially algebraic theories;
2. $\mathcal{F}_\infty$ is bi-initial in $\mathcal{L}og$, by giving any topos the *maximal* ranking. $\square$

Internalisation of syntax

Proposition (Maietti)

T *a finite ess. alg. theory. Then any* arithmetic universe *(= list-arithmetic pretopos)* $\mathcal{E}$ *has an initial model of* T, *and AU functors preserve these.* □

In particular, any topos $\mathcal{E}$ has a free internal r-topos $\mathcal{F}_r^{\mathcal{E}}$.

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In particular, any topos $\mathcal{E}$ has a free internal r-topos $\mathcal{F}_r^{\mathcal{E}}$.

In fact $\mathcal{F}_r^{\mathcal{E}}$ is built by Gödel-numbering, so admits choice principles:

Proposition

For any AU $\mathcal{E}$*,* $(\mathrm{Pred}\,\mathcal{E})_{\mathrm{ex/lex}}$ *is an AU, in which many covers split.* □

Corollary (Categorical Gödel-numbering)

1. *The initial model of* $\mathbb{T}$ *in* $\mathcal{E}$ *factors through* $(\mathrm{Pred}\,\mathcal{E})_{\mathrm{ex/lex}} \longrightarrow \mathcal{E}$.
2. *All types of the initial model are enumerable; certain covers between them split.* □

Indexed and internal categories

Framework for handling external and internal categories together: indexed categories ($\approx$ fibred categories).

Definition

An $\mathcal{E}$-indexed category: a pseudo-functor $\mathcal{E}^{\mathrm{op}} \longrightarrow \mathit{Cat}$. A $\mathcal{E}$-indexed category C is regular (resp. a topos, etc) if each C_X is regular (a topos) and each reindexing functor is regular (logical).

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Examples

- Any $\mathcal{E}$ with pullbacks has canonical self-indexing, $\mathcal{E}_X := \mathcal{E}/X$.
- An internal category C in $\mathcal{E}$ can be seen as an $\mathcal{E}$-indexed category, with $\mathrm{C}_X := \mathcal{E}(X, \mathrm{C})$.
- Any ordinary category C has the constant $\mathcal{E}$-indexing, $C_X := C$.

Concrete internal subcategories

Need to internalise both the standard interpretation $\mathcal{F} \longrightarrow \mathrm{Set}$, and Freyd gluing $\mathcal{F} \longrightarrow (\mathrm{Set} \downarrow \Gamma)$;

i.e. want logical functors $\mathcal{F}_r^{\mathcal{E}} \longrightarrow \mathcal{E}$ and $\mathcal{F}_r^{\mathcal{E}} \longrightarrow (\mathcal{E} \downarrow \Gamma)$.

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Concrete toposes come from universes;
concrete ranked toposes come from ranked universes.

Ranked universes

Definition

An *r-ranked universe* in $\mathcal{E}$ is:

- a sequence of families $U_0 \subseteq U_1 \subseteq \cdots U_r \subseteq U$ in $\mathcal{E}$,
- each closed under $\times$, 1, N, and subobjects,
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Proposition

An r-ranked universe determines an internal full sub-r-topos of $\mathcal{E}$. □

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Theorem (Reflection)

In a topos $\mathcal{E}$, for $r < \infty$,

1. *any family is contained in some r-ranked universe;*
2. *any family is contained in some internal full sub-r-topos $\mathrm{E} \subseteq \mathcal{E}$.* $\square$

Internal standard model

Proposition

For $\mathcal{E}$ topos, $r < \infty$, there's an $\mathcal{E}$-indexed r-logical functor $\mathcal{F}_r^{\mathcal{E}} \longrightarrow \mathcal{E}$.

Proof.

Take internal sub-r-topos $\mathrm{E} \hookrightarrow \mathcal{E}$; then initiality gives r-logical $\mathcal{F}_r^{\mathcal{E}} \longrightarrow \mathrm{E} \hookrightarrow \mathcal{E}$. □

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Proposition

Internalisation of syntax followed by internal interpretation is interpretation. That is, the following diagram commutes up to iso:

$$\begin{tikzcd} \mathcal{F}_r \arrow[drr, bend left, "{[\![-]\!]}"] \arrow[d, "\ulcorner - \urcorner"'] \arrow[dr, phantom, "\cong"] & & \\ \mathcal{F}_r^{\mathcal{E}} \arrow[rr, "{[\![-]\!]}"] & & \mathcal{E} \end{tikzcd}$$

□

Freyd gluing, ranked + internalised

Proposition

Ranked toposes are closed under the same gluing constructions as toposes. □

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Proposition

Let $\mathcal{E}$ be a topos, and $\mathcal{F}_r^{\mathcal{E}}$ its internal free r-topos. Then $\mathcal{E}$ believes 'the terminal object of $\mathcal{F}_r^{\mathcal{E}}$ is projective'.

Proof.

Take internal concrete r-topos $\mathrm{E} \hookrightarrow \mathcal{E}$ containing all hom-sets of $\mathcal{F}_r^{\mathcal{E}}$, so that $\Gamma : \mathcal{F}_r^{\mathcal{E}} \longrightarrow \mathcal{E}$ factors through $\Gamma' : \mathcal{F}_r^{\mathcal{E}} \longrightarrow \mathrm{E}$.

Now glue as in original Freyd argument: $(\mathrm{E} \downarrow \Gamma')$ is internal r-topos, so $\mathcal{F}_r^{\mathcal{E}}$ maps into it as needed. □

Endgame

Theorem

N is projective in $\mathcal{F}$; so RC_ω is admissible for IHOL_N.

Proof.

- Suppose $e : A \twoheadrightarrow N$ a cover in $\mathcal{F}$.
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Takehomes

- Everything can be categorialised!
- ...usually fruitfully: abstract the formal generalities, modularise the argument
- Reflection principle: existence of ranked universes in toposes
- Gödel-numbering: exact completions of arithmetic universes
- Payoff: N projective in $\mathcal{F}$ $-$ RC_ω admissible in IHOL_N
  (indeed also $\mathrm{RC}_{\mathrm{dep}}$: every total graph in $\mathcal{F}$ has some branch)

Forssell, Lumsdaine and Swan 2026, 'Makkai's lost proof of projectivity of N in the free topos', arXiv:2604.01139

Thanks!

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