Algebraic proof theory for distributive sequent calculi and GBI-algebras

Christian Naess (joint work with Nick Galatos)

University of Denver

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Substructural Logics and Residuated Lattices

A *sequent* is a pair (Γ, π), often written $\Gamma \Rightarrow \pi$, where Γ is a sequence of formulas over some language and π is a formula which is in some sense "yielded" by Γ.

A *sequent calculus* consists of rules for deriving sequents from sequents. Rules are separated into *logical rules*, which describe the behavior of the logical connectives internal to the formulas, and *structural/metalogical rules*, which describe the behavior of sequences of formulas.

Substructural logics are presented by sequent calculi which are weaker than Gentzen's sequent calculi for classical logic **LK** and intuitionistic logic **LJ**, in the sense that they lack some of the structural rules present in **LJ**.

One important example of a substructural logic is the logic presented by the *full Lambek calculus* **FL**, whose sequent calculus is obtained by removing all structural rules from Gentzen's calculus **LJ**.

An example of a structural rule is the exchange rule:

$$\frac{\Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow \pi}{\Gamma, \Sigma_2, \Sigma_1, \Delta \Rightarrow \pi}\ (e)$$

and an example of a logical rule is the join left rule:

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \pi \qquad \Gamma, \beta, \Delta \Rightarrow \pi}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \pi}\ (\vee\mathsf{L})$$

Note that the logical rule does not have any logical connectives in the upper part of the sequent. This is a property we require of logical rules so deductions in the calculus only introduce connectives. This results in a nicer backwards proof search.

Here the Γ and Δ are metavariables for sequences of formulas that are included so the active (red) part of the rule (e) may be applied in any (blue) "context". It will be convenient for us to instead package this "context" into a polynomial u. With this notation the rules (e) and $(\vee\mathsf{L})$ look like:

$$\frac{u(\Sigma_1, \Sigma_2) \Rightarrow \pi}{u(\Sigma_2, \Sigma_2) \Rightarrow \pi}\ (e) \quad \text{and} \quad \frac{u(\alpha) \Rightarrow \pi \quad u(\beta) \Rightarrow \pi}{u(\alpha \vee \beta) \Rightarrow \pi}\ (\vee\mathsf{L})$$

Substructural logics can be studied through their equivalent algebraic semantics (in the sense of Blok and Pigozzi), which are varieties (of expansions of) residuated lattices.

A residuated lattice $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ is an algebra such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, and $\mathbf{A}$ satisfies the law of residuation:

$$x \cdot y \leqslant z \iff y \leqslant x\backslash z \iff x \leqslant z/y \qquad (\cdot\text{res})$$

FL-algebras are expansions of residuated lattices by an arbitrary constant 0 in the language. The variety of FL-algebras is the equivalent algebraic semantics for the calculus **FL**.

The 0 is used to define negation ($\neg x = x\backslash 0$); however, for simplicity of this presentation our language will not have this 0 constant.

Structural rules of a sequent calculus can be interpreted as quasiequations in the language of the equivalent algebraic semantics.

Consider again the exchange rule:

$$\frac{\Gamma, \Sigma_1, \Sigma_2, \Delta \Rightarrow \pi}{\Gamma, \Sigma_2, \Sigma_1, \Delta \Rightarrow \pi}\ (e)$$

We can interpret this as the quasiequation.

$$g \cdot a \cdot b \cdot d \leqslant c \implies g \cdot b \cdot a \cdot d \leqslant c$$

Note the "$\Rightarrow$" in the rule is the sequent arrow, whereas the " $\implies$ " in the quasiequation is propositional "implies" and corresponds to the horizontal line in the rule. Further, by residuation the quasiequation can be converted to an equation as follows.

Residuate c and d on each equation to obtain the quasiequation $(a \cdot b \leqslant g\backslash c/d) \implies (b \cdot a \leqslant g\backslash c/d)$ which is equivalent to $a \cdot b \leqslant b \cdot a$.

Algebraic Proof Theory

Ciabattoni, Galatos, and Terui in their 2012 paper [CGT] developed *algebraic proof theory* in order to establish a connection between proof theoretic and algebraic approaches to the study of substructural logics in such a way that techniques of one discipline may be applied to the other.

In particular, the authors show that for extensions of **FL** by various classes of structural rules, the proof theoretic property of *strong analyticity* is equivalent to the algebraic property of *closure under (Dedekind-MacNeille) completions*.

Extending to the Distributive

The goal of this work is to extend the results of [CGT] to the setting of distributive residuated lattices and their equivalent sequent calculi.

This involves extending the sequent calculus **FL** to a distributive sequent calculus **DFL**. To see how this is done, we will attempt to add a structural rule corresponding to meet-join distributivity in the same way as exchange corresponds to the equation $xy \leqslant yx$.

Consider the non-trivial part of distributivity: $x \wedge (y \vee z) \leqslant (x \wedge y) \vee (x \wedge z)$.
This is equivalent to: $(x \wedge y) \leqslant p$ and $(x \wedge z) \leqslant p \implies x \wedge (y \vee z) \leqslant p$.
Interpreting this quasiequation as a rule in **FL** gives the following:

$$\frac{\Gamma, \sigma \wedge \alpha, \Delta \Rightarrow \pi \quad \Gamma, \sigma \wedge \beta, \Delta \Rightarrow \pi}{\Gamma, \sigma \wedge (\alpha \vee \beta), \Delta \Rightarrow \pi}$$

This is not a rule we wish to add to a calculus as it has logical connectives in the upper part of the rule.

To solve this issue, we introduce a second structural connective which corresponds to logical conjunction. Thus **DFL** (our distributive expansion of **FL**) has two structural or metalogical connectives. The first is the comma "," we had before and now we have added semicolon ";" which corresponds to $\wedge$.

Now replacing $\wedge$ by ; our distributive rule

$$\frac{\Gamma, \sigma \wedge \alpha, \Delta \Rightarrow \pi \quad \Gamma, \sigma \wedge \beta, \Delta \Rightarrow \pi}{\Gamma, \sigma \wedge (\alpha \vee \beta), \Delta \Rightarrow \pi} \quad \text{becomes} \quad \frac{\Gamma, (\sigma; \alpha), \Delta \Rightarrow \pi \quad \Gamma, (\sigma; \beta), \Delta \Rightarrow \pi}{\Gamma, \sigma; (\alpha \vee \beta), \Delta \Rightarrow \pi}$$

Packaging all the "context" $\Gamma, (\sigma; \quad), \Delta$ into the polynomial notation we can rewrite this rule as:

$$\frac{u(\alpha) \Rightarrow \pi \quad u(\beta) \Rightarrow \pi}{u(\alpha \vee \beta) \Rightarrow \pi}\ (\vee\mathrm{L})$$

Which fulfills our demand rules not have logical connectives in the upper part of the rule. In fact, this rule is the "distributive" version of the join left rule we saw earlier which is already present in **FL**; however, the contexts are now more complex as they have two structural connectives.

It can be shown that **DFL** has a conservative extension by an implication connective "$\to$" and corresponding left and right logical rules. This arrow is Heyting in nature and has the same relationship with the new structural connective ; as the divisions $\backslash$ and $/$ have to the comma , .

For this reason we consider the expansion of the calculus and hence we also expand the semantics with $\to$ and add a second residuation law which implies distributivity of the semantics.

The resulting structure is called a GBI-algebra (generalized bunched implication algebra) introduced by Galatos and Jipsen [GJ].

A GBI-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, \to, \top, 1)$ is an algebra such that $(A, \wedge, \vee, \top)$ is a lattice with greatest element $\top$, $(A, \cdot, 1)$ is a monoid and $\mathbf{A}$ is residuated with respect to $(\cdot, \backslash, /)$ and $(\wedge, \to)$ i.e., $\mathbf{A}$ satisfies two residuation laws:

$$x \cdot y \leqslant z \iff y \leqslant x \backslash z \iff x \leqslant z / y \qquad (\cdot\text{res})$$

$$x \wedge y \leqslant z \iff y \leqslant x \to z \qquad (\wedge\text{res})$$

Another reason to study GBI-algebras is that they are a generalization of BI-algebras (bunched implication algebras) which are the equivalent semantics for *bunched implication logic*.

Bunched implication logic was introduced in 1999 by Peter O'Hearn and David Pym and has applications in computer science including program verification as its notion of consequence models coevolution of resources and processes whose synchronization is constrained by available resources.

The rules for the calculus **GBI** are as follows: the $(\vee L)$ rule which, as we saw, captures distributivity is highlighted in blue.

$$\frac{\Sigma \Rightarrow \alpha \quad u(\alpha) \Rightarrow \pi}{u(\Sigma) \Rightarrow \pi}\ (\mathsf{CUT}) \qquad \frac{}{\alpha \Rightarrow \alpha}\ (\mathsf{Id}) \qquad \frac{u(\Sigma;\ (\Gamma;\ \Delta)) \Rightarrow \pi}{u((\Sigma;\ \Gamma);\ \Delta) \Rightarrow \pi}\ (;\ \mathsf{a})$$

$$\frac{u(\Sigma_1;\ \Sigma_2) \Rightarrow \pi}{u(\Sigma_2;\ \Sigma_1) \Rightarrow \pi}\ (;\ e) \qquad \frac{u(\Sigma) \Rightarrow \pi}{u(\Sigma;\ \Gamma) \Rightarrow \pi}\ (;\ \mathsf{i}) \qquad \frac{u(\Sigma;\ \Sigma) \Rightarrow \pi}{u(\Sigma) \Rightarrow \pi}\ (;\ \mathsf{c}) \qquad \frac{u(\Sigma, (\Gamma, \Delta)) \Rightarrow \pi}{u((\Sigma, \Gamma), \Delta) \Rightarrow \pi}\ (,\mathsf{a})$$

$$\frac{\Sigma \Rightarrow \alpha \quad u(\beta) \Rightarrow \pi}{u(\Sigma, (\alpha\backslash\beta)) \Rightarrow \pi}\ (\backslash\mathsf{L}) \qquad \frac{\alpha, \Sigma \Rightarrow \beta}{\Sigma \Rightarrow \alpha\backslash\beta}\ (\backslash\mathsf{R}) \qquad \frac{\Sigma \Rightarrow \alpha \quad u(\beta) \Rightarrow \pi}{u((\beta/\alpha), \Sigma) \Rightarrow \pi}\ (/\mathsf{L}) \qquad \frac{\Sigma, \alpha \Rightarrow \beta}{\Sigma \Rightarrow \beta/\alpha}\ (/\mathsf{R})$$

$$\frac{u(\alpha, \beta) \Rightarrow \pi}{u(\alpha \cdot \beta) \Rightarrow \pi}\ (\cdot\mathsf{L}) \qquad \frac{\Sigma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Sigma, \Gamma \Rightarrow \alpha \cdot \beta}\ (\cdot\mathsf{R}) \qquad \frac{u(\varepsilon) \Rightarrow \alpha}{u(1) \Rightarrow \alpha}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathsf{R})$$

$$\frac{u(\alpha) \Rightarrow \pi \quad u(\beta) \Rightarrow \pi}{u(\alpha \vee \beta) \Rightarrow \pi}\ (\vee\mathsf{L}) \qquad \frac{\Sigma \Rightarrow \alpha}{\Sigma \Rightarrow \alpha \vee \beta}\ (\vee\mathsf{R}\ell) \qquad \frac{\Sigma \Rightarrow \beta}{\Sigma \Rightarrow \alpha \vee \beta}\ (\vee\mathsf{R}r)$$

$$\frac{u(\alpha;\ \beta) \Rightarrow \pi}{u(\alpha \wedge \beta) \Rightarrow \pi}\ (\wedge\ \mathsf{L}) \qquad \frac{\Sigma \Rightarrow \alpha \quad \Sigma \Rightarrow \beta}{\Sigma \Rightarrow \alpha \wedge \beta}\ (\wedge\mathsf{R})$$

$$\frac{\Sigma \Rightarrow \alpha \quad u(\beta) \Rightarrow \pi}{u(\Sigma;\ (\alpha \to \beta)) \Rightarrow \pi}\ (\to\mathsf{L}) \qquad \frac{\Sigma;\ \alpha \Rightarrow \beta}{\Sigma \Rightarrow \alpha \to \beta}\ (\to\mathsf{R}) \qquad \frac{u(\delta) \Rightarrow \pi}{u(\top) \Rightarrow \pi}\ (\top\mathsf{L}) \qquad \frac{}{\Sigma \Rightarrow \top}\ (\top\mathsf{R})$$

The system **GBI**

Distributive Hierarchy

In order to have some control over the properties of extensions of **GBI** by structural rules we introduce the *distributive substructural hierarchy* defined similarly to the *substructural hierarchy* [CGT].

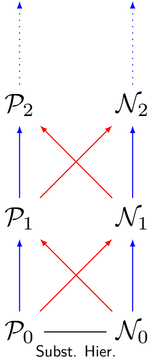
Given a residuated lattice $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$, since multiplication distributes over join, one can naturally identify a semiring reduct $\mathbf{S}_\cdot = (A, \vee, \cdot, 1)$, and a commutative semigroup reduct $\mathbf{V} = (A, \wedge)$. Further, the following equations hold:

$$s\backslash(v \wedge u) = (s\backslash v) \wedge (s\backslash u)$$
$$(r \cdot s)\backslash v = s\backslash(r\backslash v)$$
$$(s \vee r)\backslash v = (s\backslash v) \wedge (r\backslash v)$$

Thus viewing $\backslash$ as an action of $\mathbf{S}_\cdot = (A, \vee, \cdot, 1)$ on $\mathbf{V} = (A, \wedge)$ gives rise to a natural semimodule structure within $\mathbf{A}$. Similarly $/$ can be viewed as an action giving rise to a second semimodule.

Polarity

Assigning positive polarity to the connectives that produce "scalars" $(\cdot, \vee, 1)$ and negative polarity to the connectives that produce "vectors" $(\wedge, \backslash, /)$ and keeping track of their alternations in a formula, we obtain the *substructural hierarchy* pictured below.



Here $\mathcal{P}_0 = \mathcal{N}_0$ is the variable set, and the polarity of the connectives organizes the formulas into *positive formulas* $\mathcal{P}_n$ and *negative formulas* $\mathcal{N}_n$.

The formulas classes $\mathcal{P}$ and $\mathcal{N}$ also have connections to proof theory. The polarity of a logical connective can also be defined based on the invertibility of the corresponding left or right rule.

Example:
$$\frac{u(\alpha) \Rightarrow \pi \quad u(\beta) \Rightarrow \pi}{u(\alpha \vee \beta) \Rightarrow \pi}\ (\vee\mathsf{L})$$

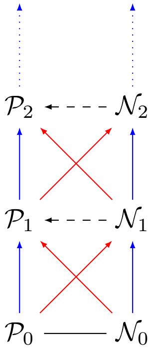
Given a GBI-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, \rightarrow, 1, \top)$, meet residuation necessitates that meet distributes over join. Thus we can identify an additional semiring reduct $\mathbf{S}_\wedge = (A, \vee, \wedge, \top)$, and $\mathbf{V} = (A, \wedge)$ remains a commutative semigroup.

Similar to before, the equations:

$$(s \vee r) \rightarrow v = (s \rightarrow v) \wedge (r \rightarrow v)$$
$$s \rightarrow (v \wedge u) = (s \rightarrow v) \wedge (s \rightarrow u)$$
$$(r \wedge s) \rightarrow v = s \rightarrow (r \rightarrow v)$$

hold in GBI-algebras, so we can now view $\rightarrow$ as a (two sided) action giving rise to a semimodule $\mathbf{S}_\wedge$ over $\mathbf{V}$.

Now following our assignment of polarity as before, $\wedge$ is also a positive connective and $\rightarrow$ is a negative connective. This gives the positive and negative connectives of the *distributive substructural hierarchy* as $\{\vee, \cdot, \wedge, 1, \top\}$ and $\{\wedge, \backslash, /, \rightarrow\}$ respectively.



Dist. Subst. Hierarchy

The formulas are classified into the *distributive substructural hierarchy* pictured to the left. As before, the formula classes $\mathcal{P}$ and $\mathcal{N}$ are determined by alternations of positive and negative connectives appearing in the construction of a formula.

The solid arrows denote inclusion and the dashed arrows denote partial inclusion (caused by $\wedge$ being both positive and negative).

The benefit of relativizing to distributive calculi is that the complexity of formulas in the hierarchy is reduced.

Examples:

(D): $x \wedge (y \vee z) \leqslant (x \wedge y) \vee (x \wedge z)$ is on the level $\mathcal{N}_3$ of the substructural hierarchy but on the level $\mathcal{N}_2$ of the distributive hierarchy.

(DMM): $x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z)$ is on the level $\mathcal{N}_3$ of the substructural hierarchy but on the level $\mathcal{N}_2$ of the distributive hierarchy.

An equation $t_0 \leqslant t_1 \vee \ldots \vee t_n$ is *simple* if t_0 has no variables repeated, and each t_i is a $\{\cdot, \wedge, 1\}$-term.

Given a simple equation (ε) of the form $t_0 \leqslant t_1 \vee \ldots \vee t_n$. We can construct a structural rule as follows:

$$\frac{u(t_1) \Rightarrow \pi \quad \ldots \quad u(t_n) \Rightarrow \pi}{u(t_0) \Rightarrow \pi}\ (R(\varepsilon))$$

Theorem: If an $\mathcal{N}_2$ equation satisfies an easy to check combinatorial condition called *acyclicity* then it is equivalent to a finite set of *simple equations.*

Dedekind-MacNeille Completions

Given an FL-algebra **A**, a completion $(\mathbf{B}, \iota)$ is called the *Dedekind-MacNeille completion* if **A** embeds as a join-dense and meet-dense subset $\iota[A] \subseteq B$ of **B** by the embedding ι.
i.e., for every $x \in B$ there are sets $A_1, A_2 \in \iota[A]$ such that $x = \bigvee A_1$ and $x = \bigwedge A_2$. Dedekind-MacNeille completions are unique up to isomorphisms fixing A so we often refer to the Dedekind-MacNeille completion. This is true of FL-algebras and so is true of GBI-algebras.

Further, a GBI-algebra embeds into its Dedekind-MacNeille completion as a GBI-algebra [GJ]. This is not generally the case for distributive residuated lattices as the Dedekind-MacNeille completion is known for not preserving distributivity.

Strong Analyticity

As previously mentioned one desirable property of a sequent calculus is a terminating backwards proof search.

An issue with the calculus **GBI** as presented is the cut rule:

$$\frac{\Sigma \Rightarrow \alpha \quad u(\alpha) \Rightarrow \pi}{u(\Sigma) \Rightarrow \pi}\ (\mathsf{CUT})$$

If a sequent is provable in a calculus **L** there is another proof of the sequent in **L**$-$(CUT), then we say **L** has *cut-elimination*.

Another property we might desire of our calculus is the *subformula property* which holds whenever a sequent s is provable in a system, there is a proof of s where any formula appearing in the proof is a subformula of s. In particular proof involving the cut rule do not satisfy this condition so the subformula property implies cut admissability.

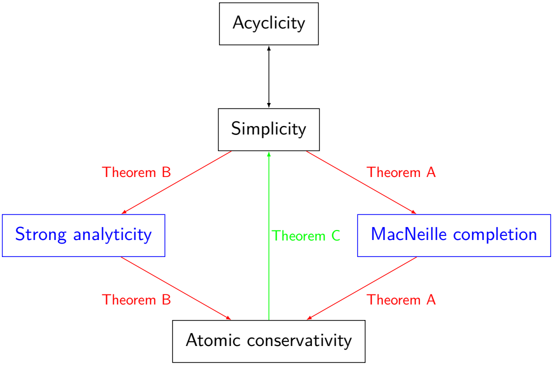
If a calculus has the subformula property for deductions from sets of *atomic sequents* (sequents consisting of atomic formulas) closed under cuts, then we say that it is *strongly analytic*.

Theorem: For any set R of structural rules, the following are equivalent:

(i) The set of (simple) equations corresponding to R is preserved by Dedekind-MacNeille completions.

(ii) R is equivalent to a set of rules R' such that $\mathbf{GBI}_{R'}$ is strongly analytic.

(iii) R is equivalent to a set of acyclic $\mathcal{N}_2$ equations.



Generalizations

- Not-necessarily associative nGBI-algebras.
- Expansion of language with arbitrary constant 0.

Future work

- Expansion of language with $\perp$ lower bound and unit of $\vee$.
- Extend results of [CGT] to involutive **FL** and involutive frames.
- Expand from sequent calculi to hypersequent calculi. This expands the level of the hierarchy that can be analyzed to a subset of the $\mathcal{P}_3$ axioms, called the $\mathcal{P}_3^\flat$ axioms.

Thank you!

Atomic Conservativity

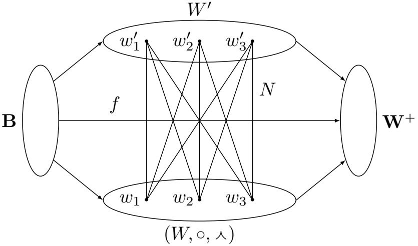
The first auxiliary concept is *atomic conservativity*.
$\mathbf{GBI}^\kappa$ is the expansion of the calculus **GBI** by rules for κ-infinitary lattice connectives. An example of such an infinitary rule:

$$\frac{u(\alpha_i) \Rightarrow \pi \text{ for some } i \in I}{u\Big(\bigwedge_{i \in I} \alpha_i \Rightarrow \pi\Big)} \ (\textstyle\bigwedge \mathsf{L})$$

Given a set R of structural rules and a cardinal κ, the system $\mathbf{GBI}^\kappa_R$ is a *conservative extension* (respectively *atomic conservative extension*) of **GBI** provided that if S is a set of sequents (resp. atomic sequents), and s is a sequent in the language of **GBI**, $S \vdash_{\mathbf{GBI}^\kappa_R} s$ implies that $S \vdash_{\mathbf{GBI}_R} s$.

Distributive Gentzen Frames

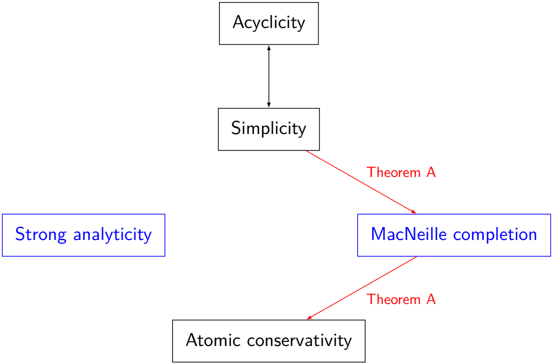
The second auxiliary concept is that of *distributive Gentzen frames* [GJ].



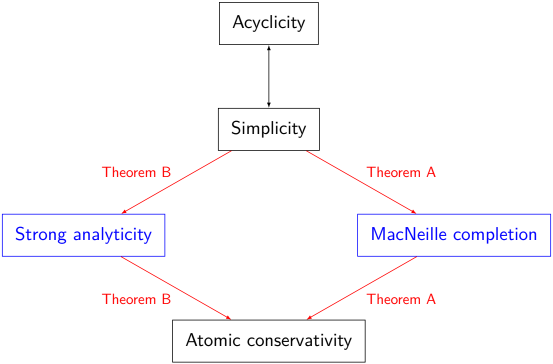
A typical Gentzen frame with the dual algebra.

These relational structures are similar to the Dedekind-MacNeille completion in that they encode some information about the structure into the frame relation N and produce the *dual algebra* or *Galois algebra* $\mathbf{W}^+$. The set $\mathbf{B}$ is in some sense common to both sorts W and W' and the map f embeds the behavior of the sequent calculus into the dual algebra. This results in an algebra (the dual algebra) of the correct type which has been constructed to witness proof theoretic properties in its semantics.

Theorem A: If E is a set of acyclic $\mathcal{N}_2$-equations/axioms, then the subvariety $\mathbf{GBI}_E$ of **GBI** admits MacNeille completions, and $\mathbf{GBI}_E^\kappa$ is a conservative extension of $\mathbf{GBI}_R$ for every cardinal κ.



Theorem B: If R is a set structural rules corresponding to simple equations, then $\mathbf{GBI}_R^\kappa$ is strongly analytic for every cardinal κ. Further If $\mathbf{GBI}_R^\kappa$ is strongly analytic, then $\mathbf{GBI}_R^\kappa$ is an atomic conservative extension of $\mathbf{GBI}_R$.



Theorem C: Let R be a set of structural rules. If $\mathbf{GBI}_R^\omega$ is an atomic conservative extension of $\mathbf{GBI}_R$, then R is equivalent to a set of analytic structural rules.