The Deducibility Problem for Certain Extensions of the Full Non-Associative Lambek Calculus is Undecidable

August Sangalli
(Joint Work With Nick Galatos)

University of Denver

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nFL is $\mathbf{FL}$ without associativity.

Residuated Lattices

A *not necessarily associative residuated lattice (nRL)* is an algebraic structure $\mathbf{R} = (R, \vee, \wedge, \cdot, \backslash, /, 1)$ so that $(R, \vee, \wedge)$ is a lattice, $(R, \cdot, 1)$ is a groupoid and the *law of residuation* holds: For all $x, y, z \in R$,

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Recall that a quasi-equation is the universal closure of implications of the following form:

$$s_1 = t_1 \ \& \ \ldots \ \& \ s_n = t_n \ \Rightarrow \ s = t$$

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Define the 1-step computation relation $\rightsquigarrow_1$ for words of length at least 2 by $a_i a_j u \rightsquigarrow_1 u\tau_i$ where $a_i, a_j \in \mathcal{T}$ and $u \in \mathcal{T}^*$.

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Fact: The halting problem for 2-tag systems is undecidable. In fact, there exists a 2-tag system for which $\mathrm{Acc}(\tau)$ is undecidable.

Examples

Let $\mathcal{T} := \{a_1, a_2, a_3\}$, $\tau_1 := a_2a_3$, $\tau_2 := a_1$, and $\tau_3 := a_1a_1$.

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In fact, $a_2a_2a_3, a_3a_1, a_1a_1, a_2a_3, a_1 \in \mathsf{Acc}(\tau)$.

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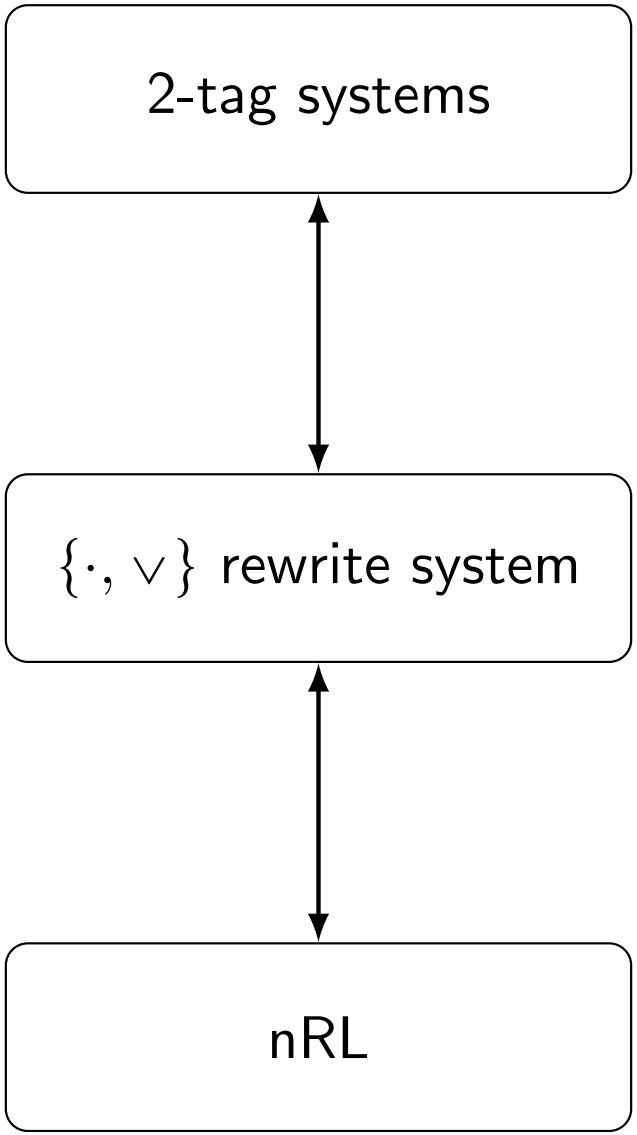
Therefore, $a_1a_2a_3 \notin \mathsf{Acc}(\tau)$.

Overview



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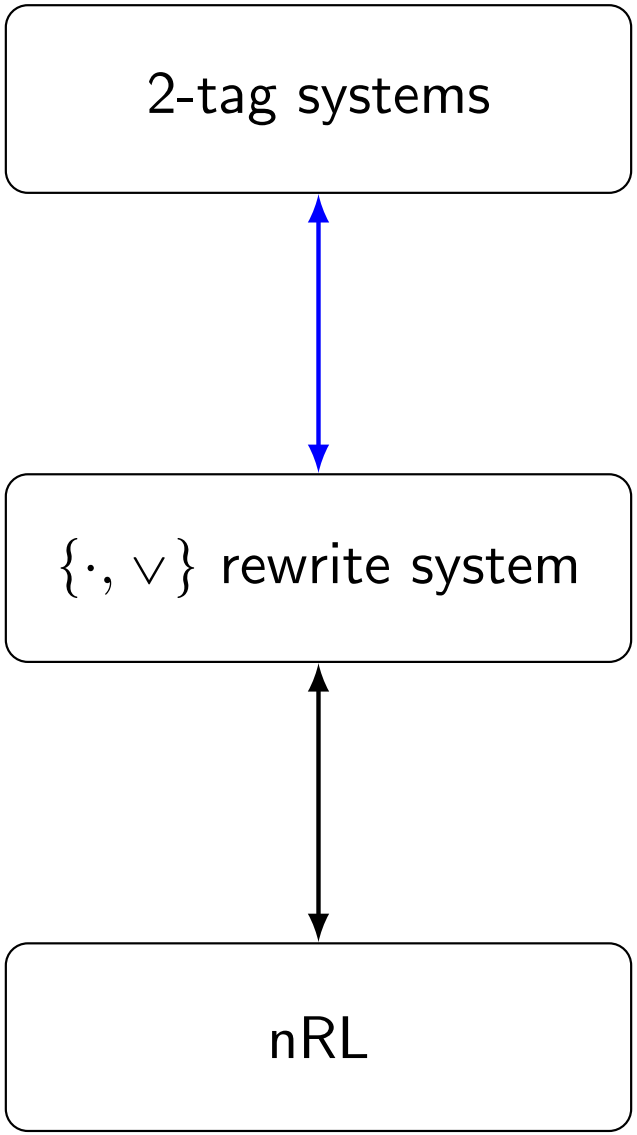

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Similarly, applying the *naive* rule $a_i^j \leqslant \varepsilon$ would allow us to delete *any* pair.

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Full $\{\cdot, \vee\}$ Rewrite System

$$w \in \mathcal{T}^* \text{ is represented by } e\overline{w}X \text{ (formally } \overrightarrow{e\overline{w}X}\text{)}$$

- To control the alterations of appending and deleting, we introduce the letters e, e' and X, X'.
- We use the convention of append first then delete.
- We introduce 30 rewrite rules (instructions) denoted by P.
- Utilize $\vee$ to branch off into parallel computations that verify a rule has been applied correctly.
- For a given 2-tag system $\tau : \mathcal{T} \to \mathcal{T}^*$ let R_τ designate our $\{\cdot, \vee\}$ rewrite system.

Correctness of Rewrite System

Recall the 2-tag system where $\tau_1 = a_2a_3, \tau_2 = a_1, \tau_3 = a_1a_1$. Note: $a_2a_3 \rightsquigarrow_1 a_1$

$$ea_2^3X$$

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Recall the 2-tag system where $\tau_1 = a_2a_3, \tau_2 = a_1, \tau_3 = a_1a_1$. Note: $a_2a_3 \rightsquigarrow_1 a_1$

$(26) : a_j^k X \leqslant \overline{a_j^k \tau_i} X' \vee a_j^k A_i$

$$ea_2^3 X \leqslant e(a_2^3 a_1 X' \vee a_2^3 A_2)$$

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$$\begin{aligned} ea_2^3 X &\leqslant e(a_2^3 a_1 X' \vee a_2^3 A_2) \\ &= ea_2^3 a_1 X' \vee ea_2^3 A_2 \end{aligned}$$

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$(26) : a_j^k X \leqslant \overline{a_j^k \tau_i} X' \vee a_j^k A_i$ $(2) : ea_i^j A_i \leqslant eA$

$$\begin{aligned} ea_2^3 X &\leqslant e(a_2^3 a_1 X' \vee a_2^3 A_2) \\ &= ea_2^3 a_1 X' \vee ea_2^3 A_2 \\ &\leqslant ea_2^3 a_1 X' \vee eA \end{aligned}$$

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Recall the 2-tag system where $\tau_1 = a_2a_3, \tau_2 = a_1, \tau_3 = a_1a_1$. Note: $a_2a_3 \rightsquigarrow_1 a_1$

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$$\begin{aligned} ea_2^3 X &\leqslant e(a_2^3 a_1 X' \vee a_2^3 A_2) \\ &= ea_2^3 a_1 X' \vee ea_2^3 A_2 \\ &\leqslant ea_2^3 a_1 X' \vee eA \\ &\leqslant e(e' \vee d')a_1 X' \vee eA \end{aligned}$$

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$$\begin{aligned} ea_2^3 X &\leqslant e(a_2^3 a_1 X' \vee a_2^3 A_2) \\ &= ea_2^3 a_1 X' \vee ea_2^3 A_2 \\ &\leqslant ea_2^3 a_1 X' \vee eA \\ &\leqslant e(e' \vee d')a_1 X' \vee eA \\ &= ee' a_1 X' \vee ed' a_1 X' \vee eA \end{aligned}$$

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$(10) : X' \leqslant D'$

$$\begin{aligned} ea_2^3 X &\leqslant e(a_2^3 a_1 X' \vee a_2^3 A_2) \\ &= ea_2^3 a_1 X' \vee ea_2^3 A_2 \\ &\leqslant ea_2^3 a_1 X' \vee eA \\ &\leqslant e(e' \vee d')a_1 X' \vee eA \\ &= ee' a_1 X' \vee ed' a_1 X' \vee eA \\ &\leqslant ee' a_1 X' \vee ed' a_1 D' \vee eA \end{aligned}$$

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$$
\begin{aligned}
ea_2^3 X &\leqslant e(a_2^3 a_1 X' \vee a_2^3 A_2) \\
&= ea_2^3 a_1 X' \vee ea_2^3 A_2 \\
&\leqslant ea_2^3 a_1 X' \vee eA \\
&\leqslant e(e' \vee d') a_1 X' \vee eA \\
&= ee' a_1 X' \vee ed' a_1 X' \vee eA \\
&\leqslant ee' a_1 X' \vee ed' a_1 D' \vee eA \\
&\leqslant ee' a_1 X' \vee ed' D' \vee eA
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$$

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$$
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&\leqslant ea_2^3 a_1 X' \vee eA \\
&\leqslant e(e' \vee d')a_1 X' \vee eA \\
&= ee'a_1 X' \vee ed'a_1 X' \vee eA \\
&\leqslant ee'a_1 X' \vee ed'a_1 D' \vee eA \\
&\leqslant ee'a_1 X' \vee ed'D' \vee eA \\
&\leqslant ee'a_1 X' \vee eA \vee eA
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$$
\begin{aligned}
ea_2^3X &\leqslant e(a_2^3a_1X' \vee a_2^3A_2) \\
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$$
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&\leqslant ea_2^3 a_1 X' \vee eA \\
&\leqslant e(e' \vee d')a_1 X' \vee eA \\
&= ee'a_1 X' \vee ed'a_1 X' \vee eA \\
&\leqslant ee'a_1 X' \vee ed'a_1 D' \vee eA \\
&\leqslant ee'a_1 X' \vee ed'D' \vee eA \\
&\leqslant ee'a_1 X' \vee eA \vee eA \\
&\leqslant ee'X' \vee eA \vee eA \leqslant eX \vee eA \vee eA
\end{aligned}
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&\leqslant e(e' \vee d')a_1 X' \vee eA \\
&= ee'a_1 X' \vee ed'a_1 X' \vee eA \\
&\leqslant ee'a_1 X' \vee ed'a_1 D' \vee eA \\
&\leqslant ee'a_1 X' \vee ed'D' \vee eA \\
&\leqslant ee'a_1 X' \vee eA \vee eA \\
&\leqslant ee'X' \vee eA \vee eA \leqslant eX \vee eA \vee eA \Rightarrow ea_2^3 X \in \mathsf{Acc}(R_\tau)
\end{aligned}
$$

Rewrite System Does Not Over Compute

Consider the 2-tag system where $\tau_1 := a_3$, $\tau_2 := a_2$, and $\tau_3 := a_3a_3a_3$.

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Correct derivation:

$\underline{a_1}a_2 \rightsquigarrow_1 \cancel{a_1 a_2} a_3$ which implies $a_1a_2 \in \mathsf{Acc}(\tau)$.

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Double appending glitch:

$\underline{a_1}a_2$

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$\underline{a_1}a_2 \rightsquigarrow_1 a_1a_2\underline{a_3}$

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$\underline{a_1}a_2 \rightsquigarrow_1 a_1a_2\underline{a_3} \rightsquigarrow a_1a_2a_3a_3\underline{a_3}a_3$

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Detection in rewrite system:

$$ea_1^2X$$

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Detection in rewrite system:

$(26) : a_j^k X \leqslant \overline{a_j\tau_i}X' \vee a_j^k A_i$

$$ea_1^2X \leqslant ea_1^2a_3X' \vee ea_1^2A_1$$

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$$\begin{aligned} ea_1^2 X &\leqslant ea_1^2 a_3 X' \vee ea_1^2 A_1 \\ &\leqslant ea_1^2 a_3^3 a_3^3 X \vee ea_1^2 a_3 A_1' \vee ea_1^2 A_1 \end{aligned}$$

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$\underline{a_1}a_2 \rightsquigarrow_1 a_1a_2\underline{a_3} \rightsquigarrow a_1a_2a_3a_3\underline{a_3}a_3 \rightsquigarrow_1 \cancel{a_1a_2a_3a_3}\underline{a_3}a_3 \rightsquigarrow_1 \ldots$

Detection in rewrite system:

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$(13) : a_j^k a_l A_i' \leqslant a_j^k A_i'$

$$\begin{aligned} ea_1^2X &\leqslant ea_1^2a_3X' \vee ea_1^2A_1 \\ &\leqslant ea_1^2a_3^3a_3^3X \vee ea_1^2a_3A_1' \vee ea_1^2A_1 \\ &\leqslant ea_1^2a_3^3a_3^3X \vee ea_1^2A_1' \vee ea_1^2A_1 \end{aligned}$$

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$\underline{a_1}a_2 \rightsquigarrow_1 a_1a_2\underline{a_3} \rightsquigarrow a_1a_2a_3a_3\underline{a_3}a_3 \rightsquigarrow_1 \cancel{a_1a_2a_3a_3}\underline{a_3}a_3 \rightsquigarrow_1 \ldots$

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$$\begin{aligned} ea_1^2 X &\leqslant ea_1^2 a_3 X' \vee ea_1^2 A_1 \\ &\leqslant ea_1^2 a_3^3 a_3^3 X \vee ea_1^2 a_3 A_1' \vee ea_1^2 A_1 \\ &\leqslant ea_1^2 a_3^3 a_3^3 X \vee ea_1^2 A_1' \vee ea_1^2 A_1 \leqslant ?? \end{aligned}$$

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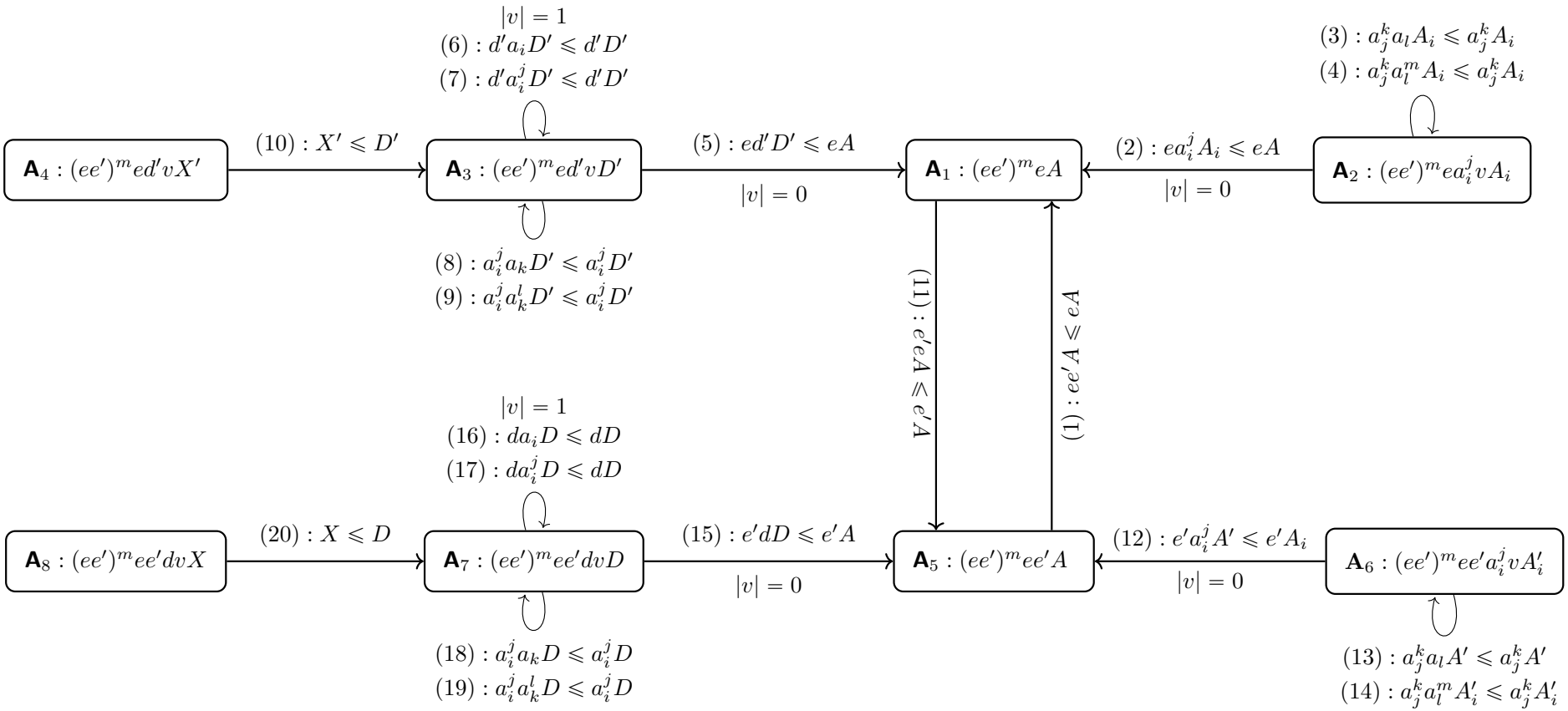
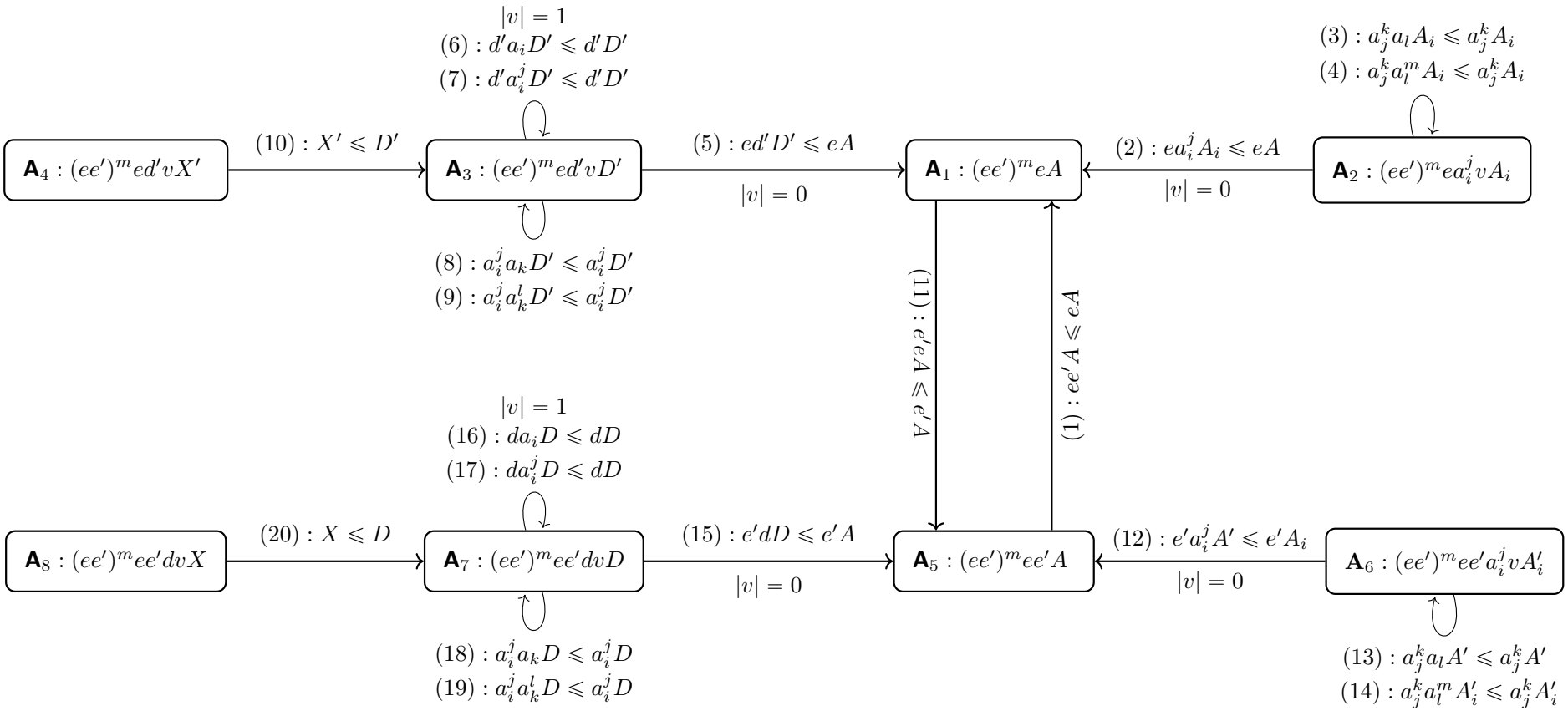
$$\begin{aligned} ea_1^2 X &\leqslant ea_1^2 a_3 X' \vee ea_1^2 A_1 \\ &\leqslant ea_1^2 a_3^3 a_3^3 X \vee ea_1^2 a_3 A_1' \vee ea_1^2 A_1 \\ &\leqslant ea_1^2 a_3^3 a_3^3 X \vee ea_1^2 A_1' \vee ea_1^2 A_1 \leqslant ?? \end{aligned}$$

Similarly, the auxiliary configurations from the deletion rules $(29) : a_i^j \leqslant e' \vee d'$ and $(30) : a_i^j \leqslant e \vee d$ prohibit deleting before appending.

Rewrite System Successfully Simulates Accepted Words

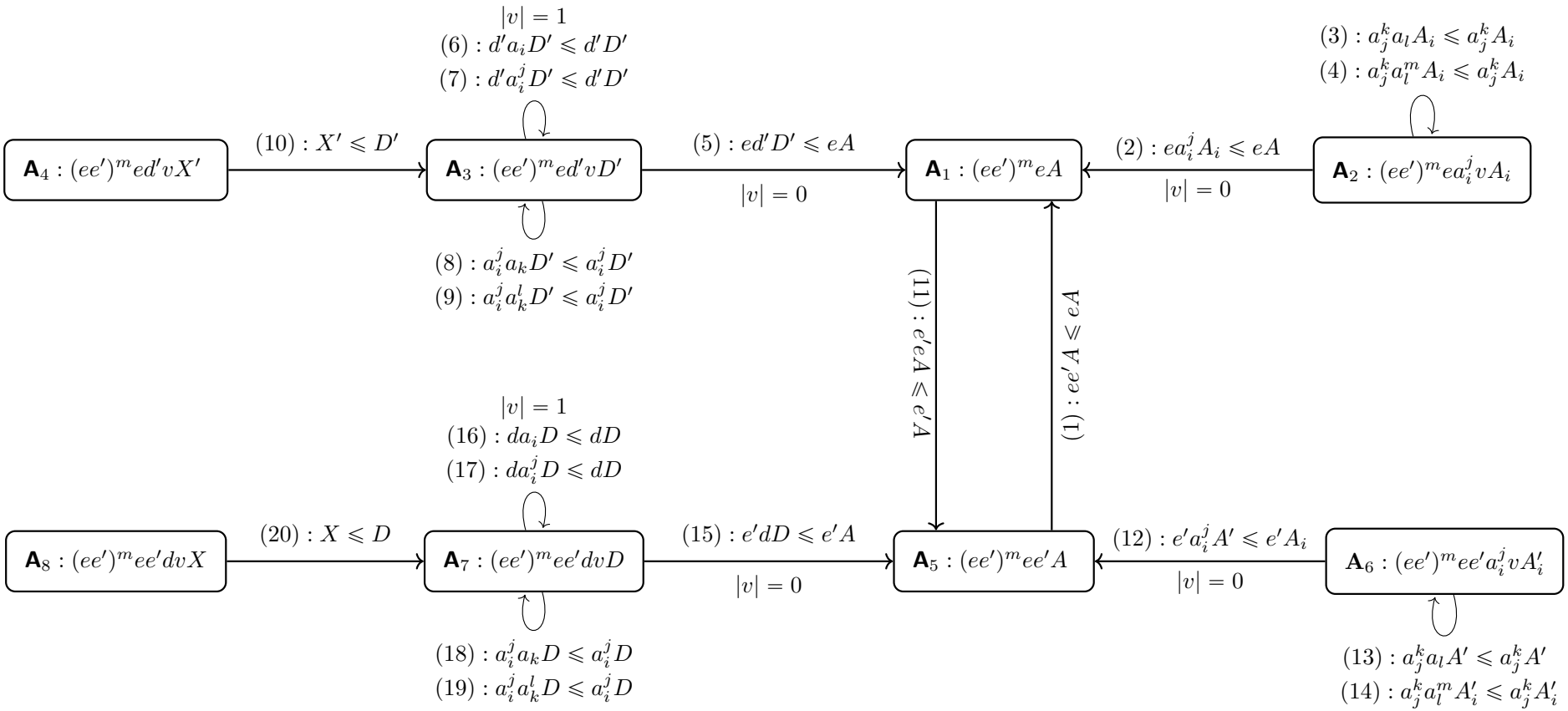


Rewrite System Successfully Simulates Accepted Words



$$ea_2^3X \leqslant \underbrace{ea_2^3a_1X'}_{\text{Main}} \vee \underbrace{ea_2^3A_2}_{\text{Aux:}\mathbf{A}_2}$$

Rewrite System Successfully Simulates Accepted Words



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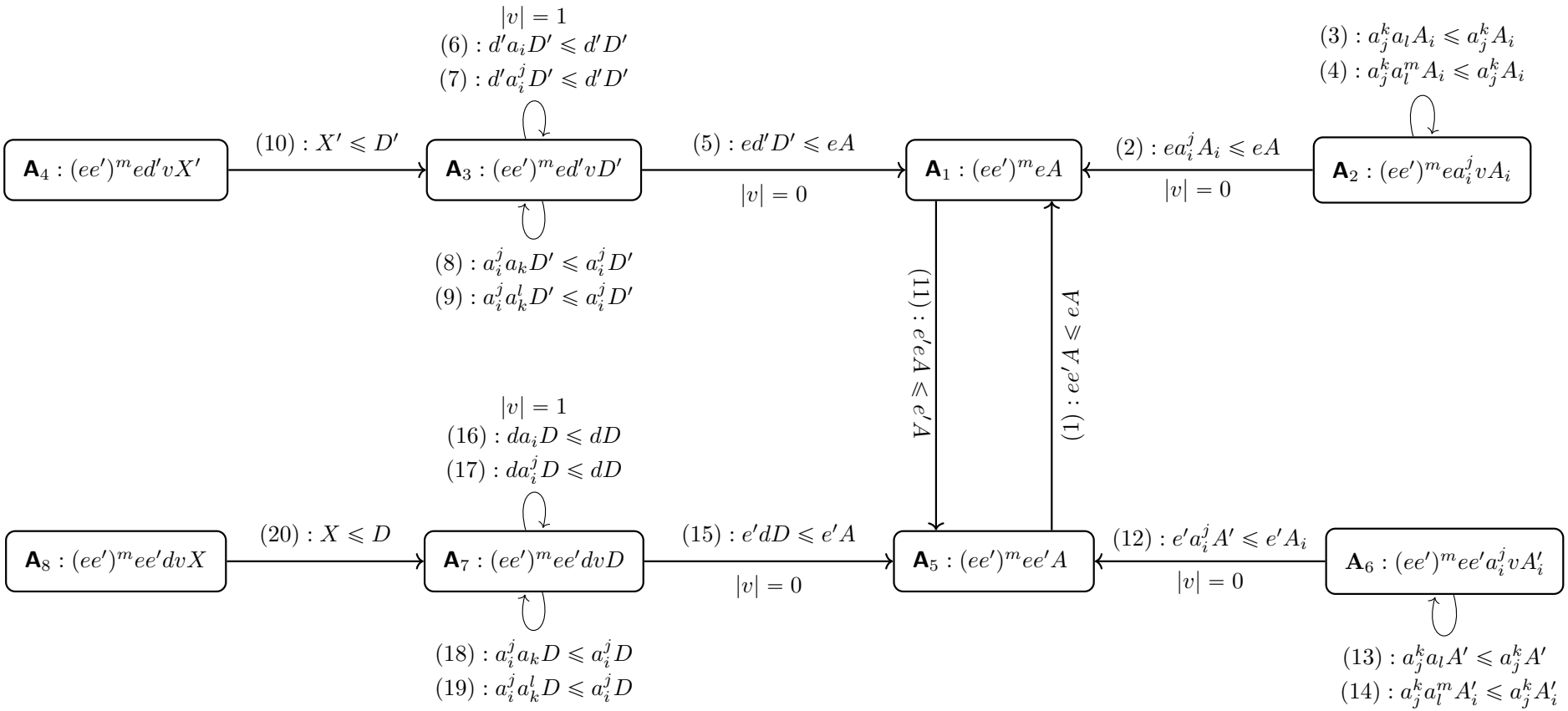

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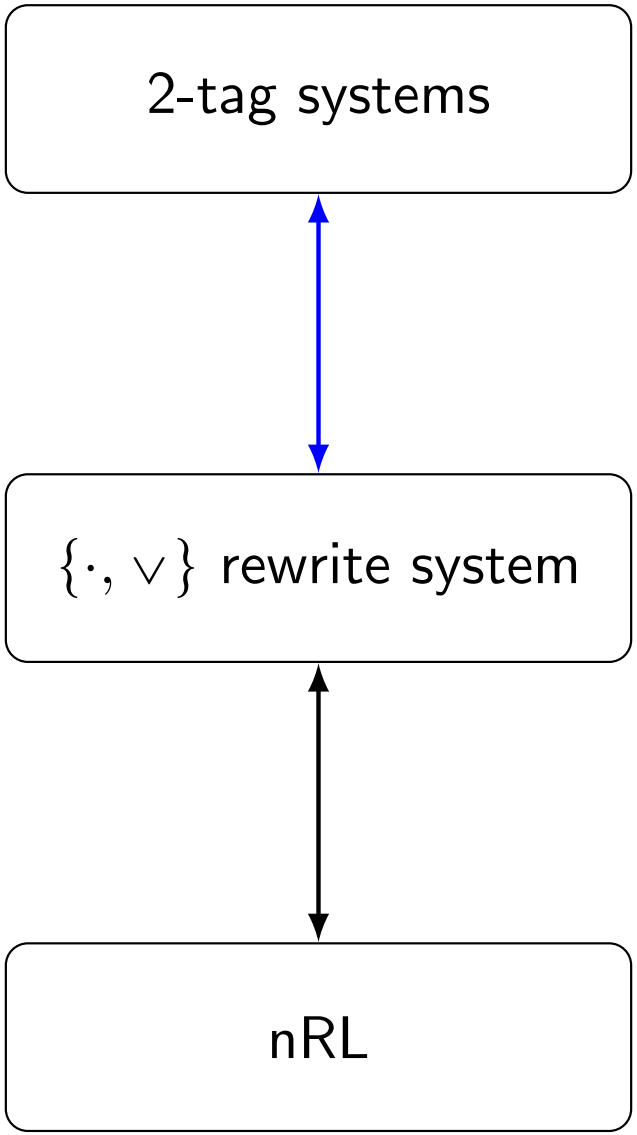
Fact: Let $\tau : \mathcal{T} \to \mathcal{T}^*$ be a 2-tag system: $w \in \mathsf{Acc}(\tau)$ iff $\overrightarrow{e\overline{w}X} \in \mathsf{Acc}(R_\tau)$.

Overview



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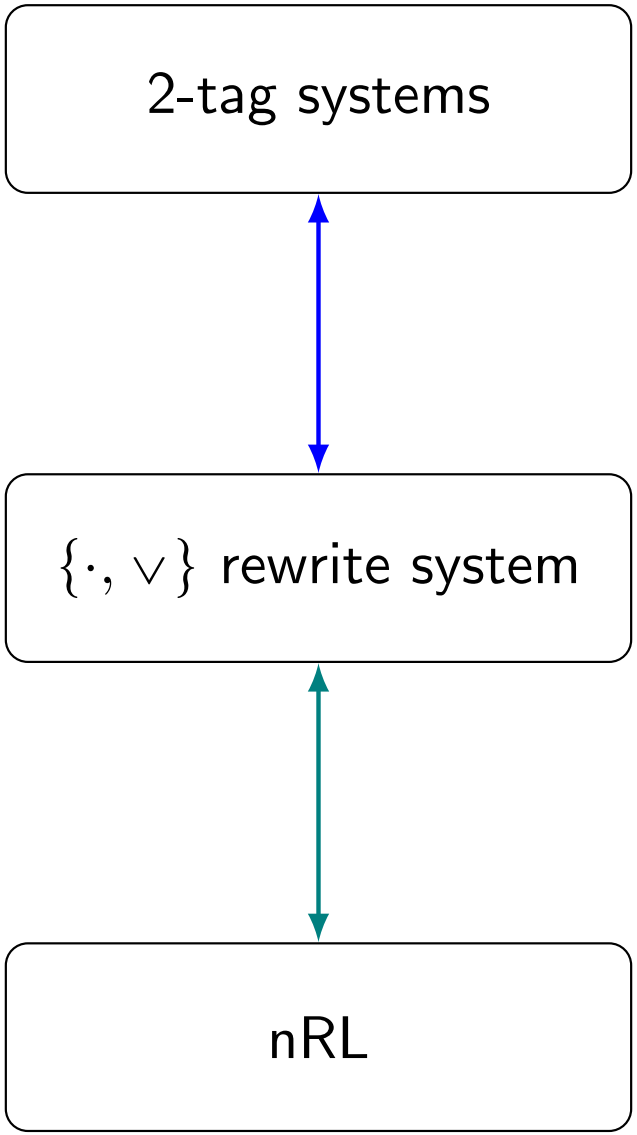

Main Result

Theorem: Let $\mathcal{V}$ be an variety in the interval $[\mathsf{RL}, \mathsf{nRL}]$ and $\tau : \mathcal{T} \to \mathcal{T}^*$ a 2-tag system. Then for each $w \in \mathcal{T}^*$:
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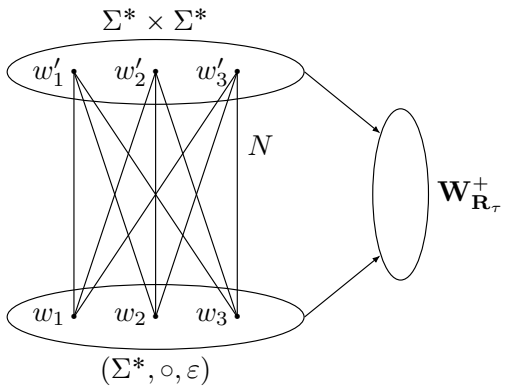

Corollary: Any variety in [RL, nRL] has an undecidable word problem/quasi-eq. theory. (Any logic in $[\mathbf{nFL}, \mathbf{FL}]$ has an undecidable deducibility problem)

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- What other structural rules can be added while maintaining undecidability? (E.g. knotted rules, simple equations, etc)
- Can we obtain undecidability of the equational theory for certain subvarieties of nRL?

Thank You!!