Constructive Quantum Logics

Guillaume Massas

Chapman University

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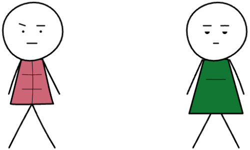

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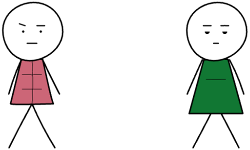

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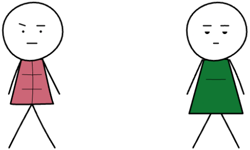

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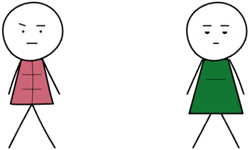

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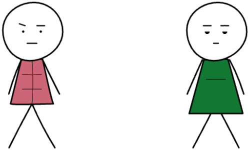

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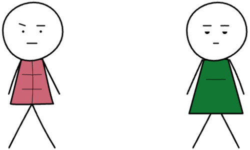

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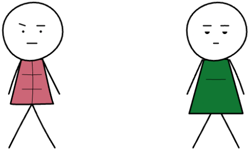

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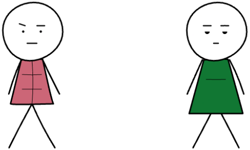

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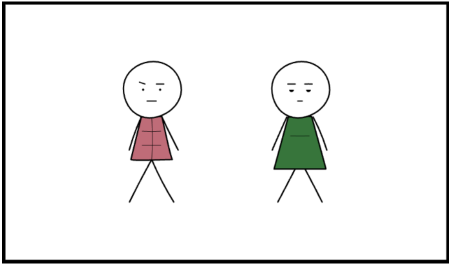
- One more caveat: Because we do not always have implications, we think of a logic as a binary consequence relation on the set of formulas, not as a set of theorems.

Fundamental Logic



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- Fundamental logic is the logic of fundamental lattices, bounded lattices L equipped with a unary operation $\neg : L \to L$ satisfying:
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  - $a \leq \neg b$ implies $b \leq \neg a$ (dual self-adjointness);
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- It is also the logic determined by the intersection of the set of inference rules valid in IPC^- and the set of those valid in OL in Fitch's natural deduction system.

$$
\begin{tikzcd}
 & \mathsf{CPC} & \\
\mathsf{O} \arrow[ur, "\mathrm{Reit}"] & & \mathsf{IPC}^{-} \arrow[ul, "\mathrm{RAA}"'] \\
 & \mathsf{F} \arrow[ul, "\mathrm{RAA}"] \arrow[ur, "\mathrm{Reit}"'] &
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- She can also understand orthologic via the Goldblatt translation, which fully and faithfully embeds OL into KTB (the modal logic of reflexive and symmetric frames).
- Motivating idea: use non-classical modal logics to understand the relationship between fundamental logic, constructive reasoning "from the viewpoint of" the quantum logician, and quantum reasoning "from the viewpoint of" the constructive logician.

- Heloise can try to understand Abelard by considering the orthomodal logic of reflexive and transitive frames OS4, and the preimage of the GMT translation;
- Abelard can try to understand Heloise by considering the intuitionistic modal logic of reflexive and symmetric frames FSTB, and the preimage of the Goldblatt translation.

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Theorem (Holliday and M., 2026)

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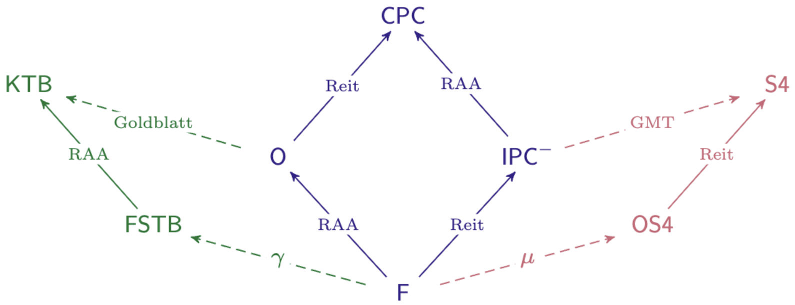
- An extension of this result to the language $\{\wedge, \vee, \rightarrow\}$ was recently obtained by Zhicheng Chen, *Fundamental Propositional Logic with Preconditional* (2026).


CPC
Reit
RAA
KTB
S4
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O
IPC⁻
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γ
μ
F

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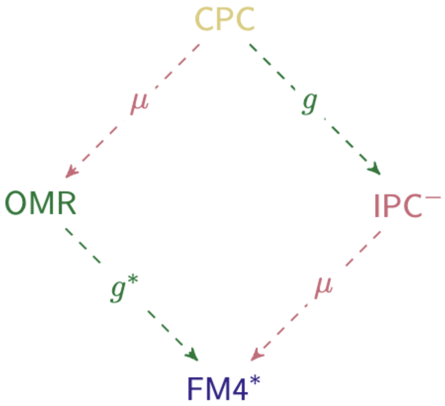
Theorem (Holliday and M., 2026)

There exists a fundamental logic $\mathsf{FM4}^*$ *and an orthomodal logic* OMR *such that:*

- *The GMT translation* μ *embeds* IPC^- *into* $\mathsf{FM4}^*$*;*

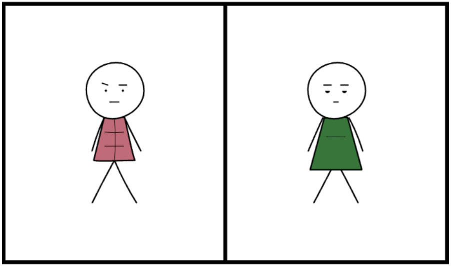
- *The GMT translation* μ *embeds* CPC *into* OMR*;*

- *A natural modal extension of the double negation translation* g *embeds* OMR *into* $\mathsf{FM4}^*$*.*



- More on this: Holliday and M., *Fundamental Logic through the Lens of Modality*, preprint.

Ex Logic



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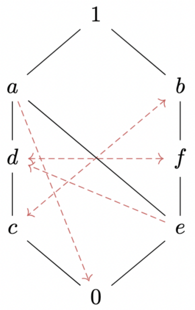
- Fact: $\neg\neg a \wedge \neg\neg b \leq \neg\neg(a \wedge b)$ is not valid in fundamental logic.



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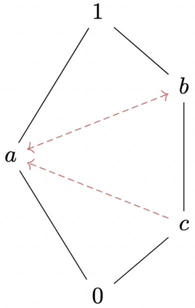

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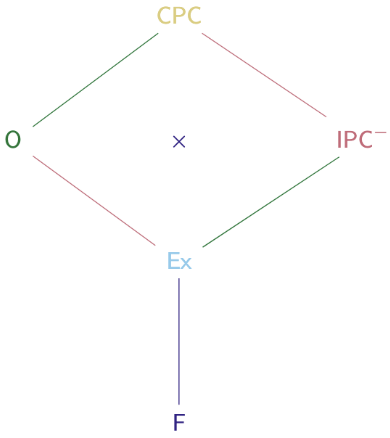
Theorem (Aguilera and M., 2026)

The intersection of IPC^{-} and OL is axiomatized by adding a single, 6-variable axiom (Ex) to fundamental logic.

Moreover, the lattice of extensions of Ex is isomorphic to the product of the lattice of extensions of IPC^{-} and the lattice of extensions of OL.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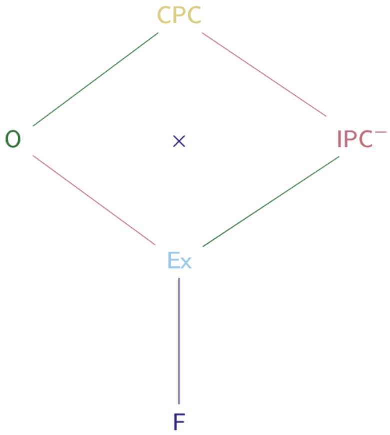

- More on this: Aguilera and M., *Constructive Quantum Logics*, Proc. of the Roy. Soc. A, 2026.

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- So the question becomes: what is the intersection of IPC and OML?

- Algebraically, the intersection of IPC and OML is the logic of the join variety $\mathbb{V}$ of the variety **HA** of Heyting algebras and the variety **OMA** of orthomodular algebras (orthomodular lattices with the Sasaki implication in the signature).

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- **Key Idea**: find some inequations to impose on a fundamental lattice L so that:
  1. the Heyting implication and the Sasaki implication both satisfy these inequations;
  2. the inequations guarantee that L can be embedded into the product of a HA and an OMA.

An Ex-algebra is a bounded lattice L equipped with a binary operation $\to$ such that:

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&\text{(iCl)} && [(a \wedge ((b \wedge c) \vee (b \wedge d))) \to g] \wedge a \leq [(b \wedge (c \vee d)) \vee ((b \wedge (c \vee d)) \to g)].
\end{aligned}
$$

$$\begin{tikzcd}
 & & L_O \\
L \arrow[urr, two heads, "\cdot_O"] \arrow[r, hook] \arrow[drr, two heads, "\cdot_I"'] & L_O \times L_I \arrow[ur, two heads] \arrow[dr, two heads] & \\
 & & L_I
\end{tikzcd}$$

Theorem (iEx-Embedding Theorem)

Let L be a fundamental algebra. Then L is the subdirect product of an orthomodular algebra O_L and a Heyting algebra I_L.

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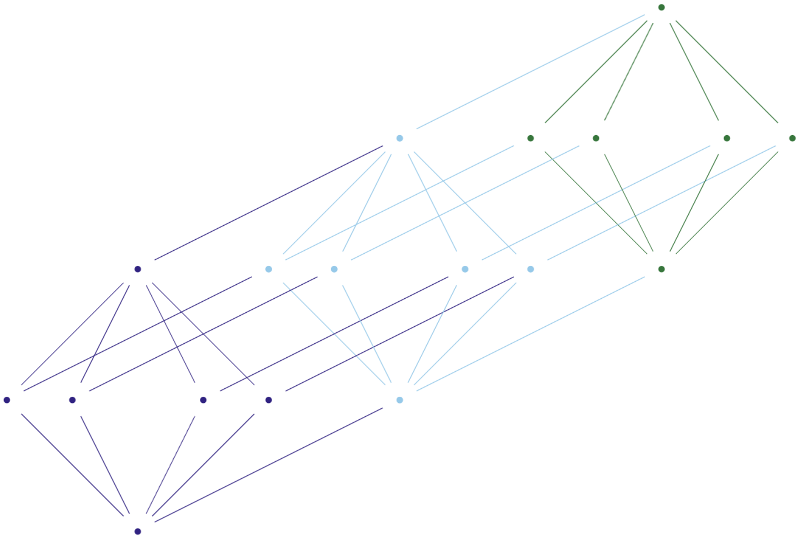
- The Heyting algebra L_I is the subalgebra of the downsets of $(P_L, \supseteq)$ (the poset of prime filters on L, ordered by converse inclusion) generated by sets of the form:

$$\widehat{a} = \{p \in P_L \mid a \in p\}.$$

“Minimal” Non-Trivial Example of an Ex-Algebra



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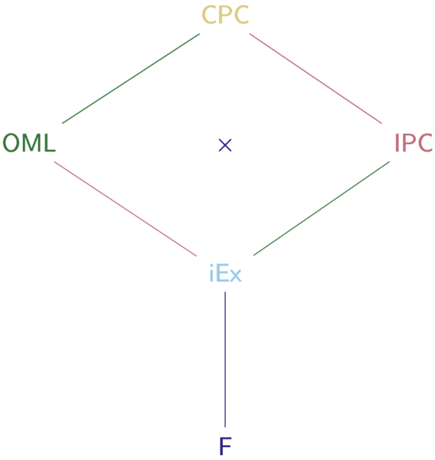
- This immediately gives that iEx, the logic of Ex-algebras, is exactly $\mathsf{IPC} \cap \mathsf{OML}$.
- From iEx, one can recover IPC by adding the axiom $a \leq b \to a$, and OML by adding $\neg\neg a \leq a$.
- In the $\{\wedge, \vee, \neg\}$ fragment, the axiomatization simplifies to the Ex axiom, and recovers $\mathsf{IPC}^{-} \cap \mathsf{OL}$.

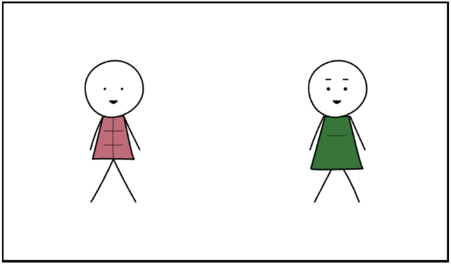
- In fact, the method is much more general: for any superintuinionistic logic L_I and any superorthomodular logic L_O, one can axiomatize in a straightforward way the logic $\mathsf{L}_I \cap \mathsf{L}_O$.

Theorem

Let $i\mathbf{SI}$ be the lattice of superintuitionistic logics, $i\mathbf{SO}$ the lattice of superorthomodular logics, and $i\mathbf{SE}$ the lattice of implicative constructive quantum logics (i.e., extensions of iEx*). Then $i\mathbf{SE} \simeq i\mathbf{SO} \times i\mathbf{SI}$.*



- More on this: Aguilera and M., *Conditionals and Modalities in Constructive Quantum Logics*, Advances in Modal Logic, Vol.16.



Thank You!