Maximal Sublattices of Finite Semidistributive Lattices

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Maximal sublattices

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A lattice is **join-semidistributive** ($\mathrm{SD}_\vee$) if:

$$x \vee y = x \vee z \implies x \vee (y \wedge z) = x \vee y$$

for all $x, y, z \in \mathcal{L}$. **Meet-semidistributive lattices** ($\mathrm{SD}_\wedge$) are defined dually.

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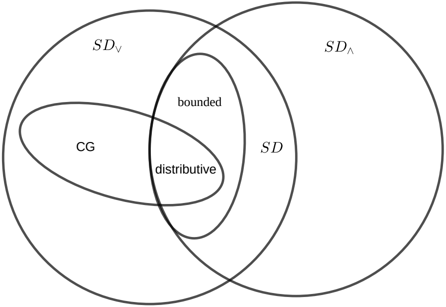
A lattice is **semidistributive** (SD) if it is both join- and meet-semidistributive.

General picture



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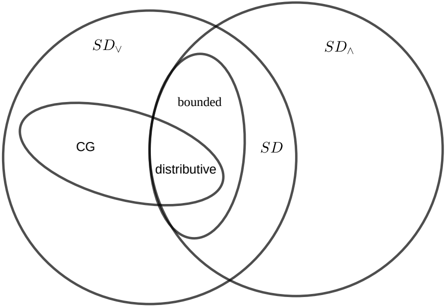

Bounded lattices = bounded homomorphic images of free lattices.

R. Freese, J. Ježek, J.B. Nation, *Free Lattices*, AMS, 1995. Chapter II

Distributive lattices

The complements of maximal sublattices of distributive lattices are always intervals of the form

$$[a, b] = \{x \in L \ : \ a \leq x \leq b\}$$

with a being a unique join-irreducible element, and b being a unique meet-irreducible one in the interval.

C.C. Chen, K.M. Koh, S.K. Tan, *Frattini sublattices of distributive lattices*, Algebra Universalis **3** (1973), 294–303.

I. Rival, *Maximal Sublattices of Finite Distributive Lattices*, Proc. Amer. Math. Soc. **37** (1973), 417–420.

Bounded lattices

The complements of maximal sublattices of bounded lattices are intervals.

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Hypothesis

The complements of maximal sublattices of:

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The complements of maximal sublattices of:

- (finite) SD lattices are intervals;
- (finite) $SD_\vee$ lattices are unions of intervals with a common minimal element.

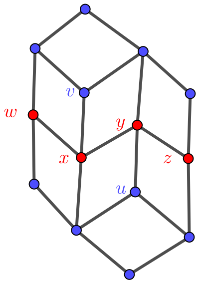
Outside of SD

This lattice is neither $\mathrm{SD}_\vee$:

$$y = u \vee x = u \vee z, \quad \text{but} \quad u \vee (x \wedge z) = u \vee 0 = u \neq u \vee x,$$

nor $\mathrm{SD}_\wedge$:

$$x = v \wedge w = v \wedge y, \quad \text{but} \quad v \wedge (w \vee y) = v \wedge 1 = v \neq v \wedge w.$$



The set $\{w, x, y, z\}$ is the complement of a maximal sublattice.

General observations

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- maximal elements in $\mathcal{C}$ are meet-irreducible and minimal elements are join-irreducible.
- $\mathcal{C}$ cannot be partitioned into two nonempty disconnected sets.

Canonical join representation

R. Freese, J. Ježek, J.B. Nation, *Free Lattices*, AMS, 1995.

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- All elements in the canonical join representation are join irreducible;
- We will refer to an element of the canonical join representation of an element $x \in \mathcal{L}$ as a **canonical joinand** of x;
- A finite lattice $\mathcal{L}$ satisfies $\mathrm{SD}_{\vee}$ if and only if every element of $\mathcal{L}$ has a canonical join representation.

$\mathrm{SD}_\vee$ lattices

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Theorem (AMSZ-D)

Suppose that $\mathcal{S}$ is a sublattice of $\mathcal{L}$, that m is maximal in $\mathcal{S} \setminus 1$ and that $c \in (m, 1)$ is a coatom. Then for any $d \in \mathcal{L}$, $c \in \langle \mathcal{S} \cup d \rangle$ if and only if $m \vee d = c$ and hence $d \leq c$. Thus, in this case if also $\mathcal{S}$ is a maximal sublattice, then c is the unique maximal element of $\mathcal{C}$.

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If there is a greatest element in $\mathcal{C}$, then $\mathcal{C}$ is an interval.

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The dual statements hold for $\mathrm{SD}_\wedge$ lattices.

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If there is an element $a \in \mathcal{C}$ that is comparable to every element of $\mathcal{C}$, then $\mathcal{C}$ is an interval.

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Lemma (AMSZ-D)

Every $x \in \mathcal{C}$ has at least one strict canonical joinand.

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Lemma (AMSZ-D)

Assume that there are $x, y, z, w \in \mathcal{L}$ such that $w \leq x \leq y \leq z$, and x is a canonical joinand for z and y is a canonical meetand for w. Then $\mathcal{L} \setminus [x, y]$ is a sublattice of $\mathcal{L}$.

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Assume additionally that x is a strict canonical joinand for z.

Then $y = z$ and $\mathcal{C} = [x, y]$.

Dually, if y is a strict canonical meetand of w, then $w = x$ and $\mathcal{C} = [x, y]$.

Illustration

The corollary says that whenever the complement of a maximal sublattice, $\mathcal{C}$ has a linear chain whose (strict) joinand/meetand relationship is "spiraling out", as illustrated below, then $\mathcal{C}$ must be an interval.

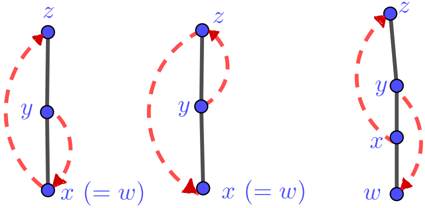
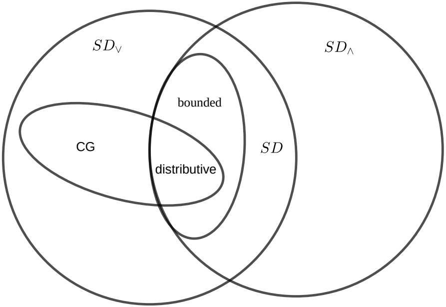


Figure: A dotted arrow from point *a* to point *b* means that *a* is a canonical joinand or a canonical meetand of *b*.

General picture



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- (finite) $\mathrm{SD}_\vee$ lattices are unions of intervals with a common minimal element.

Convex geometries

A **convex geometry** (CG) is an $SD_\vee$ lattice which is also **lower semi-modular**:

$$\forall\, x, y \in \mathcal{G}\,(x \prec x \vee y \implies x \wedge y \prec y)\,,$$

where $a \prec b$ means a is a subcover of b.

Due to results of Edelman and Jamison a CG can be represented via intersections of down-sets of chains formed on the base set of a geometry, and hence the examples can be produced easily, unlike semidistributive lattices.

We proved that a complement of maximal sublattices of convex geometries generated by just two chains is either an interval or a union of two intervals with the common minimal element.

References

AMSZ-D:

K. Adaricheva, A. Mata, S. Silberger, A. Zamojska-Dzienio, *Conjecture on Maximal Sublattices of Finite Semidistributive Lattices and Beyond*, Discrete Mathematics and Theoretical Computer Science, 28(2) #32(2026).

Thank you!