Modal Measurable Logics via a Modal Loomis-Sikorski Representation Theorem

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Develop modal logics to reason about measurable spaces.

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The resulting infinitary modal logics may provide a reasoning tool for ergodic theory.

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$$B \cong \Sigma(X)/I.$$

σ-Modal Algebras

Definition. A σ-modal algebra is a pair $(B, \Diamond)$ consisting of a σ-Boolean algebra B and a unary operator $\Diamond : B \to B$ that satisfies

$$\Diamond\bot = \bot \quad \text{and} \quad \bigvee_{n\in\omega} \Diamond b_n = \Diamond \bigvee_{n\in\omega} b_n \tag{1}$$

for any countable sequence $(b_n)_{n\in\omega}$ of elements in B.

σ-Jónsson-Tarski

Definition. A Stone space X is called a σ-Stone space if for any countable sequence $(a_n)_{n\in\omega}$ of clopen sets,

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Definition. A σ-modal space is a modal space $\mathbb{X} = (X, \tau, R)$ such that (X, τ) is a σ-Stone space and for any countable sequence $(a_n)_{n\in\omega}$ of clopen sets,

$$\Diamond_R\Big(\overline{\bigcup_{n\in\omega} a_n}\Big) = \overline{\Diamond_R\Big(\bigcup_{n\in\omega} a_n\Big)}. \tag{3}$$

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Theorem (σ-Jónsson-Tarski). Every σ-modal algebra $(B, \Diamond)$ is isomorphic to $(\mathsf{Clop}(X), \overline{\bigcup}_\omega, \mathsf{int} \bigcap_\omega, \Diamond_R)$ of a modal σ-space X of its ultrafilters.

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We would like to work with σ-algebras of sets, where countable meets and joins are countable unions and intersections.

Modal measurable algebras

Definition. A measurable σ-modal algebra is a σ-modal algebra $(B, \Diamond)$ that satisfies the following condition: for every countable descending chain $(b_n)_{n\in\omega}$ (i.e. every countable collection of elements in B such that $b_0 \geq b_1 \geq b_2 \geq \cdots$),

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This rule dually corresponds to $\Diamond_R$ sending Baire null-sets to Baire null-sets, which *en route* ensures our Modal Loomis-Sikorski theorem.

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Definition. A modal measurable space is a tuple $\mathfrak{F} = (X, R, \Sigma)$ consisting of a measurable space (X, Σ) and a relation R such that for every $a \in \Sigma$ we have $\Diamond_R a := \{x \in X \mid R[x] \cap a \neq \emptyset\} \in \Sigma$.

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We call Σ a σ-modal algebra of sets.

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$$(B, \Diamond) \cong \mathrm{Baire}(\mathbb{X})/\mathcal{M}.$$

Modal measurable semantics

Definition. A marked modal measurable space or mmm-space is a tuple (X, R, Σ, N) where

- (X, R, Σ) is a modal measurable space; and
- N is a modal σ-ideal on $\mathfrak{F}^+ = (\Sigma, \Diamond_R)$ such that for all countable collections of measurable sets $(a_n)_{n \in \omega}$,

$$\bigcap_{n \in \omega} a_n \in N \quad \text{implies} \quad \bigcap_{n \in \omega} \Diamond_R(a_n) \in N.$$

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We will interpret our formulas in $(\Sigma, \Diamond_R)/N$.

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Let $f : X \to X$ be a measure-preserving transformation, as in dynamical systems and ergodic theory, i.e. $f^{-1}(b) \in \Sigma$ and $\mu f^{-1}(b) = \mu(b)$ for all $b \in \Sigma$.

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Let R be the graph of f, i.e. xRy iff $f(x) = y$, and let N be the collection of null-sets.

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Furthermore, for any sequence $(a_n)_{n\in\omega}$ of measurable sets whose intersection is in N, we have

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Therefore (X, R, Σ, N) is an mmm-space.

As a concrete example, we may consider $(X, \mathcal{O})$ to be the unit interval with the Euclidean topology, μ the Lebesgue measure and $f(x)$ the Bernoulli map $f(x) = 2x \bmod 1$.

Modal measurable semantics

Pavlov (2020) states that “One can argue that an object of the right category of spaces in measure theory is not a set equipped with a σ-algebra of measurable sets, but rather a set S equipped with a σ-algebra M of measurable sets and a σ-ideal N of negligible sets, i.e., sets of measure 0.”

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They further observe: "This 'point-free' approach to ergodic theory seems particularly well suited for studying actions of uncountable (discrete) groups, as by deleting the null-sets in advance, one can avoid to a large extent the standard difficulty that an uncountable union of null-sets is null."

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Also Scott (2009), Lando (2012), Fernández-Duque (2010).

Modal measurable logics

The modal measurable language is the collection $\mathcal{L}_\sigma$ of formulas defined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \bigvee_{i\in\omega} \varphi_i \mid \Diamond\varphi$$

where p ranges over a countably infinite set Prop of proposition letters.

Modal measurable logics

Definition. The σ-modal logic K_σ is the least set of $\mathcal{L}_\sigma$-formulas derivable from the following axioms and rules:

Modal measurable logics

- All substitution instances of classical tautologies (allowing for countable conjunction and disjunction, this includes the infinite De Morgan laws);
- The join rule: for any countable sequence $(\varphi_n)_{n\in\omega}$ and formula ψ:

$$\frac{\varphi_n \to \psi \qquad n \in \omega}{(\bigvee_{n\in\omega}\varphi_n) \to \psi}$$

- The modal axiom and rule:

$$\Diamond(\bigvee_{n\in\omega}\varphi_n) \leftrightarrow \bigvee_{n\in\omega}\Diamond\varphi_n \qquad \frac{\neg\varphi}{\neg\Diamond\varphi}$$

- Modus ponens:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Modal measurable logics

Modal Loomis-Sikorski logic MLS is the least set of $\mathcal{L}_\sigma$-formulas derivable from the axioms and rules above together with the infinite descending chain rule (IDC):

$$\frac{\neg\bigwedge_{n\in\omega}\varphi_n \qquad \varphi_{n+1}\to\varphi_n \text{ for all } n\in\omega}{\neg\bigwedge_{n\in\omega}\Diamond\varphi_n}. \tag{IDC}$$

Modal measurable completeness

Theorem. The logic $\mathsf{MLS} \oplus \mathrm{Ax}$ is sound and complete with respect to the class of mmm-spaces validating Ax. That is, is for every formula φ we have

$$\mathsf{MLS} \oplus \mathrm{Ax} \vdash \varphi \quad \text{iff} \quad \mathtt{MMM}(\mathrm{Ax}) \Vdash \varphi.$$

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The key step in the proof is given by the Modal LS-Theorem.

Conclusions and future work

Modal Loomis-Sikorski theorem: Every measurable σ-modal algebra is the quotient of a "concrete" σ-modal algebra by a modal σ-ideal.

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Completeness theorem: Every axiomatic extension of Modal Loomis-Sikorski Logic is sound and complete with respect to a class of marked modal measurable spaces.

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Modal Loomis-Sikorski theorem: Every measurable σ-modal algebra is the quotient of a "concrete" σ-modal algebra by a modal σ-ideal.

Completeness theorem: Every axiomatic extension of Modal Loomis-Sikorski Logic is sound and complete with respect to a class of marked modal measurable spaces.

Further generalizations and applications of this approach.

Thank you!