Biquasiintuitionistic logic and related structures

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Plan:

I. [E–K'26] **Structural investigations in Vakarelov logics/algebras**:

1) biquasiintuitionistic and Akchurin logics/algebras;
2) generalised Reyes–Zolfaghari adjoint modal couples;
3) generalised Zarycki–Lawvere paraconsistent border operator;
4) generalised Kolmogorov–Glivenko translation.

II. [E–K'26] **Structure of the lattice $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ of clopen spectral subpresheaves over an orthocomplemented lattice L**:

1) characterisation of full paraconsistency of Brouwer and coquasiintuitionistic algebra reducts of $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$;
2) reconstruction of L by the generalised Kolmogorov–Glivenko translation;
3) adjunctions between spectral and cospectral lattices, and a multitude of induced negations;
4) $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ as a $(3,3)$-Akchurin algebra with 16 generalised Reyes–Zolfaghari adjoint modal couples.

III. [K'24'25'26] **Some applications to**:

1) causal structure of lorentzian spacetimes;
2) lattices of *- and W*- factor subalgebras;
3) algebraic quantum field theory.

Vakarelov algebras [Vakarelov'76]

- A Vakarelov algebra [Vakarelov'76] := a lattice $(K, \leq, \wedge, \vee)$ equipped with a map ${}^\dagger : K \to K$ s.t. if $x \leq y$, then $y^\dagger \leq x^\dagger$ $\forall x, y \in K$.
- Equivalently: $x^\dagger \wedge y^\dagger \geq (x \vee y)^\dagger$ $\forall x, y \in K$; $x^\dagger \vee y^\dagger \leq (x \wedge y)^\dagger$ $\forall x, y \in K$.
- Consider additional conditions on †: $\forall x, y, z \in K$:
  - n1) $x \leq x^{\dagger\dagger}$;
  - n2) $x^{\dagger\dagger} \leq x$;
  - n3) $x^\dagger \wedge y^\dagger \leq (x \vee y)^\dagger$;
  - n4) $x^\dagger \vee y^\dagger \geq (x \wedge y)^\dagger$;
  - n5) $x \wedge x^\dagger \leq y$;
  - n6) $y \leq x \vee x^\dagger$;
  - n7) if $x \wedge y \leq z$, then $x \wedge z^\dagger \leq y^\dagger$;
  - n8) if $x \leq y \vee z$, then $z^\dagger \leq x^\dagger \vee y$;
  - n9) $1^\dagger = 0$;
  - n10) $0^\dagger = 1$.
- A Vakarelov algebra will be called:
  - (resp., co)regular iff n3 (resp., n4) holds;
  - (resp., co)quasiminimal iff and n1 (resp., n2) holds;
  - (resp., co)normal iff it is bounded and n9 (resp., n10) holds;
  - (resp., co)prequasiintuitionistic iff it is bounded and n1 and n5 (resp., n2 and n6) hold;
  - (resp., co)quasiintuitionistic iff it is distributive and (resp., co)prequasiintuitionistic;
  - (resp., co)preintuitionistic iff it is (resp., co)quasiintuitionistic and n7 (resp., n8) holds.[1]
- We denote a regular (resp., coregular) † as ${}^\circ$ (resp., ${}^\bullet$).

[1] a preintuitionistic algebra = a pseudocomplemented distributive lattice [Ribenboim'49].

Heying and Brouwer algebras [Skolem'1917]

- A bounded lattice $(K, \leq, \wedge, \vee, 0, 1)$ is called:
  - a Heyting algebra [Skolem'1917, Ogasawara'39, Birkhoff'42, Certaine'43] iff it is equipped with $\rightarrow : K \times K \rightarrow K$ s.t. there is an adjunction $\cdot \wedge x \dashv x \rightarrow \cdot$ $\forall x \in K$, i.e.
  $$x \wedge y \leq z \text{ iff } x \leq y \rightarrow z \ \ \forall x, y, z \in K;$$
  - a Brouwer algebra [Skolem'1917, McKinsey–Tarski'46] iff it is equipped with $\rightarrowtail : K \times K \rightarrow K$ s.t. there is an adjunction $\cdot \rightarrowtail x \dashv x \vee \cdot$ $\forall x \in K$, i.e.
  $$x \rightarrowtail y \leq z \text{ iff } x \leq y \vee z \ \ \forall x, y, z \in K.$$
- Any of the above conditions implies distributivity of $(K, \leq, \wedge, \vee)$ [Skolem'1917].
- If $(K, \leq, \wedge, \vee, 0, 1)$ is a Heyting (resp., Brouwer) algebra, then
$$\neg x := x \rightarrow 0 \text{ (resp., } \llcorner x := 1 \rightarrowtail x) \ \ \forall x \in K.$$
- Heyting (resp., Brouwer) algebra is a (resp., co)preintuitionistic algebra w.r.t. $\neg$ (resp., $\llcorner$).
- If $(K, \leq, \wedge, \vee, ^\dagger)$ is a complete (resp., co)preintuitionistic algebra, then it is a Heyting (resp., Brouwer) algebra iff $(\bigvee_{i \in I} y_i) \wedge x = \bigvee_{i \in I}(y_i \wedge x)$ (resp., $(\bigwedge_{i \in I} y_i) \vee x = \bigwedge_{i \in I}(y_i \vee x)$) $\forall x, y_{i \in I} \in K$.

Bi-Vakarelov, biquasiintuitionistic, and Akchurin algebras

- $(K, \leq, \wedge, \vee, ^{\dagger_1}, ^{\dagger_2})$ will be called a bi-Vakarelov algebra iff $(K, \leq, \wedge, \vee, ^{\dagger_1})$ and $(K, \leq, \wedge, \vee, ^{\dagger_2})$ are Vakarelov algebras.
- $(K, \leq, \wedge, \vee, 0, 1, ^{\circ}, ^{\bullet})$ will be called a biquasiintuitionistic algebra iff $(K, \leq, \wedge, \vee, 0, 1, ^{\circ})$ is a quasiintuitionistic algebra and $(K, \leq, \wedge, \vee, 0, 1, ^{\bullet})$ is a coquasiintuitionistic algebra [Engel–Kostecki'26].
- biregular, binormal, biquasiminimal, etc., algebras are defined analogously.
- $(K, \leq, \wedge, \vee, 0, 1)$ will be called a Skolem algebra iff it is a Heyting algebra and a Brouwer algebra [Skolem'1917, Moisil'42, Rauszer'70].
- $(K, \leq, \wedge, \vee, 0, 1, ^{\circ}, ^{\bullet})$ will be called an Akchurin algebra iff $(K, \leq, \wedge, \vee, 0, 1, ^{\circ}, ^{\bullet})$ is a biquasiintuitionistic algebra and $(K, \leq, \wedge, \vee, 0, 1)$ is a Skolem algebra [Engel–Kostecki'26].

Lattice logics [Skordev'69, Schulte Mönting'73, Goldblatt'74, Vakarelov'76]

- Propositional language $\mathcal{L}$:

*– only on the next slide

  - set L of elementary propositions;
  - propositional constants: $\bot$, $\top$;
  - unary connectives: $\underset{\circ}{\sim}^{*}$, $\underset{\bullet}{\sim}^{*}$;
  - binary connectives: $\wedge$, $\vee$, $\rightarrow^{*}$, $\succ\!\!-^{*}$.

- Formulæ $\mathrm{Frm}(\mathcal{L})$: $p, q, r, \ldots$
- Sequents $\mathrm{Seq}(\mathcal{L})$: ordered pairs (p, q), denoted by $p \vdash q$.
- Consider axioms and rules:

  a1) $p \vdash p$;

  a2) $p \wedge q \vdash p$;

  a3) $p \wedge q \vdash q$;

  a4) $p \vdash p \vee q$;

  a5) $q \vdash p \vee q$;

  a6) $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$;

  a7) $p \vdash \top$;

  a8) $\bot \vdash p$;

  r1) if $p \vdash q$ and $q \vdash r$, then $p \vdash r$;

  r2) if $p \vdash q$ and $p \vdash r$, then $p \vdash q \wedge r$;

  r3) if $p \vdash r$ and $q \vdash r$, then $p \vee q \vdash r$.

- The subset of $\mathrm{Seq}(\mathcal{L})$ consisting of a1...a5 (resp., a1...a6) and all sequents obtained from them by r1...r3 is called a bounded (resp., bounded distributive) lattice logic $\mathbf{Lat}^{\bot,\top}$ (resp., $\mathbf{Lat}_{\mathrm{d}}^{\bot,\top}$).

Vakarelov's logics [Vakarelov'76]; *: also in [Goldblatt'74]; **: only in [Drobyshevich'18]

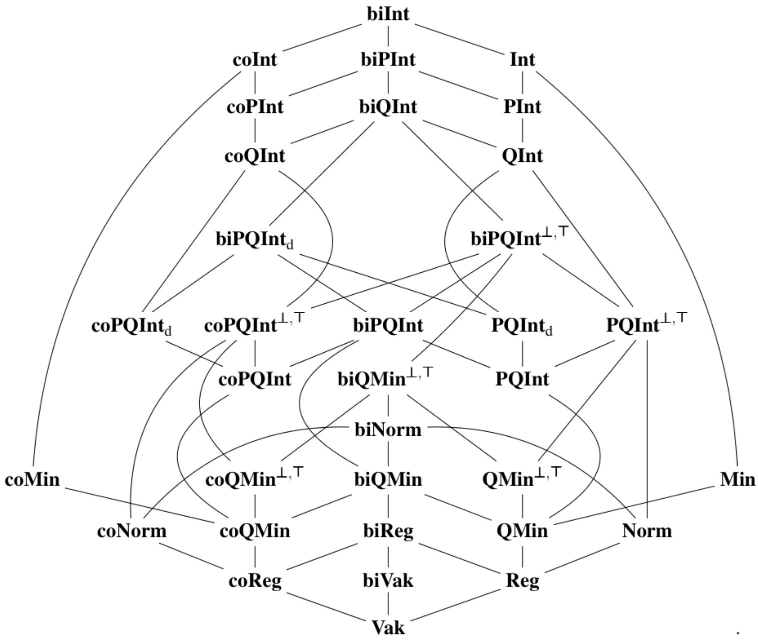
- Let

r4)*	if $p \vdash q$, then $\underset{\circ}{\sim} q \vdash \underset{\circ}{\sim} p$;	r5)*	if $p \vdash q$, then $\underset{\bullet}{\sim} q \vdash \underset{\bullet}{\sim} p$;
a9)*	$p \vdash \underset{\circ}{\sim}\underset{\circ}{\sim} p$;	a10)*	$\underset{\bullet}{\sim}\underset{\bullet}{\sim} p \vdash p$;
a11)	$\underset{\circ}{\sim} p \wedge \underset{\circ}{\sim} q \vdash \underset{\circ}{\sim}(p \vee q)$;	a12)	$\underset{\bullet}{\sim}(p \wedge q) \vdash \underset{\bullet}{\sim} p \vee \underset{\bullet}{\sim} q$;
a13)*	$p \wedge \underset{\circ}{\sim} p \vdash q$;	a14)	$q \vdash p \vee \underset{\bullet}{\sim} p$;
r6)	$p \wedge q \vdash \bot$ iff $p \vdash \underset{\circ}{\sim} q$;	r7)	$\top \vdash p \vee q$ iff $\underset{\bullet}{\sim} p \vdash q$;
a17)	$\top \vdash \underset{\circ}{\sim} \bot$;	a18)	$\underset{\bullet}{\sim} \top \vdash \bot$;
a19)	$p \wedge (p \rightarrow q) \vdash q$;	a20)**	$q \vdash p \vee (q \mathrel{\succ\!\!-} p)$;
r8)	$p \wedge q \vdash r$ iff $p \vdash q \rightarrow r$;	r9)**	$p \vdash q \vee r$ iff $p \mathrel{\succ\!\!-} q \vdash r$.

- $\mathbf{Lat}^{\bot,\top}$+(r4 or r5) =: Vakarelov logic **Vak** [Vakarelov'76];
- $\mathbf{Lat}^{\bot,\top}$+r4+a9 (resp.,+r5+a10) =: (resp., co)quasiminimal logic (resp., **co**)**QMin**;
- $\mathbf{Lat}^{\bot,\top}$+r4+a13+a17 (resp., +r5+a14+a18) =: (resp., co)normal logic (resp., **co**)**Norm**;
- **Lat**+r4+a9+a13 (resp., +r5+a10+a14) =: (resp., co)prequasiintuitionistic logic (resp., **co**)**PQInt**;
- $\mathbf{Lat}_{\mathrm{d}}^{\bot,\top}$+r4+a9+a13 (resp., +r5+a10+a14) =: (resp., co)quasiintuitionistic logic (resp., **co**)**QInt**;
- (resp., **co**)**QInt**+r6 (resp., r7) =: (resp., co)preintuitionistic logic (resp., **co**)**PInt**;
- $\mathbf{Lat}_{\mathrm{d}}^{\bot,\top}$+a19+r8 (resp.,+a20+r9) =: (resp., co)intuitionistic logic (resp., **co**)**Int**;
- **Log**+**coLog** =: **biLog** for **Log** $\in$ {**QMin**, **Norm**, **PQInt**, **QInt**, **PInt**, **Int**};
- **biQInt**+**biInt** =: Akchurin logic **biQInt** $\otimes$ **biInt** [Engel–Kostecki'26];
- **Int**: [Glivenko'29, Heyting' 30]; **biInt**: [Moisil'42, Rauszer'70]; (**co**)**PInt**: [Vakarelov'76]; $\mathbf{QMin}_{\mathrm{d}}^{\bot,\top}$: [Dunn'96]; **coQInt**: [Shramko'03]; **PQInt**: [Holliday'23].

Algebras of logic

- A valuation χ from $\mathrm{Frm}(\mathcal{L})$ in an algebra $(K, \leq, \wedge, \vee, 0, 1, ^\circ, ^\bullet)$ s.t.:
  - $\chi(p \wedge q) = \chi(p) \wedge \chi(q)$;
  - $\chi(p \vee q) = \chi(p) \vee \chi(q)$;
  - $\chi(\top) = 1$;
  - $\chi(\bot) = 0$;
  - $\chi(\underset{\circ}{\sim} p) = (\chi(p))^\circ$;
  - $\chi(\underset{\bullet}{\sim} p) = (\chi(p))^\bullet$;
  - $\chi(p \rightarrow q) = \chi(p) \rightarrow \chi(q)$;
  - $\chi(p \succ\!\!- q) = \chi(p) \succ\!\!- \chi(q)$.
- $p \vdash q \in \mathrm{Seq}(\mathcal{L})$ is valid in $(K, \leq, \wedge, \vee, 0, 1, ^\circ, ^\bullet)$ iff $\chi(p) \leq \chi(q)$ for all valuations χ of $\mathrm{Frm}(\mathcal{L})$.
- A logic **Log** is sound and complete w.r.t. the class `Alg` of algebras $(K, \leq, \wedge, \vee, 0, 1, ^\circ, ^\bullet)$ iff ($p \vdash q \in$ **Log** iff $p \vdash q \in \mathrm{Seq}(\mathcal{L})$ is valid for all members of `Alg`).
- [Schulte Mönting'73, Vakarelov'76]: $\mathbf{Lat}^{\bot,\top}$ (resp., $\mathbf{Lat}_{\mathrm{d}}^{\bot,\top}$) is sound and complete w.r.t. all bounded (resp., bounded and distributive) lattices.
- [Vakarelov'76]: general framework for proofs of soundness and completeness for Vakarelov logics (and their extensions) in terms of varieties of Vakarelov algebras (and their subvarieties).
- [Engel–Kostecki'26]: **QInt** (resp., **coQInt**; **biQInt**; **biQInt**$\otimes$**biInt**) is sound and complete w.r.t. the class of all quasiintuitionistic (resp., coquasiintuitionistic; biquasiintuitionistic; Akchurin) algebras.



biInt
coInt
biPInt
Int
coPInt
biQInt
PInt
coQInt
QInt
biPQInt_d
biPQInt^⊥,⊤
coPQInt_d
coPQInt^⊥,⊤
biPQInt
PQInt_d
PQInt^⊥,⊤
coPQInt
biQMin^⊥,⊤
PQInt
biNorm
coMin
coQMin^⊥,⊤
biQMin
QMin^⊥,⊤
Min
coNorm
coQMin
biReg
QMin
Norm
coReg
biVak
Reg
Vak

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Positive and negative modal operators

[Jónsson–Tarski'48, Vakarelov'76,'80,'89, Sotirov'78'84, Došen'84,'87, Orłowska–Vakarelov'05]:

- If $(K, \leq, \wedge, \vee, 0, 1)$ is a bounded lattice and $\Box, \Diamond, \not\Box, \not\Diamond : K \to K$, then:
  $\Box$ is a necessity operator iff $\Box(x \wedge y) = \Box x \wedge \Box y$ $\forall x, y \in K$ and $\Box 1 = 1$;
  $\Diamond$ is a possibility operator iff $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$ $\forall x, y \in K$ and $\Diamond 0 = 0$;
  $\not\Box$ is a nonnecessity operator iff $\not\Box(x \wedge y) = \not\Box x \vee \not\Box y$ $\forall x, y \in K$ and $\not\Box 1 = 0$;
  $\not\Diamond$ is a impossibility operator iff $\not\Diamond(x \vee y) = \not\Diamond x \wedge \not\Diamond y$ $\forall x, y \in K$ and $\not\Diamond 0 = 1$.
- Hence: normal (resp., conormal) logic = lattice logic with impossibility (resp., nonnecessity) operator.

[Engel–Kostecki'26]:

- $(K, \leq, \wedge, \vee, \not\Diamond_1, \ldots, \not\Diamond_n, \not\Box_1, \ldots, \not\Box_m, \Diamond_1, \ldots, \Diamond_p, \Box_1, \ldots, \Box_q) =:$ an (n, m, p, q)-Vakarelov–Sotirov algebra.
- $(K, \leq, \wedge, \vee, 0, 1, \not\Diamond_1, \ldots, \not\Diamond_n, \not\Box_1, \ldots, \not\Box_m)$ such that $(K, \leq, \wedge, \vee, 0, 1)$ is a Skolem algebra, $(K, \leq, \wedge, \vee, 0, 1, \not\Diamond_i)$ is a quasiintuitionistic algebra $\forall i \in \{1, \ldots, n\}$, and $(K, \leq, \wedge, \vee, 0, 1, \not\Box_j)$ is a coquasiintuitionistic algebra $\forall j \in \{1, \ldots, m\}$ =: an (n, m)-Akchurin algebra.
- An $(n+1, m+1, p, q)$-Vakarelov–Sotirov algebra that is an (n, m)-Akchurin algebra will be called an (n, m, p, q)-Akchurin algebra.

Reyes–Zolfaghari adjoint modal operators [Zolfaghari'92, Reyes–Zolfaghari'96]

- Given a countably additively complete Skolem algebra $(K, \leq, \wedge, \vee, 0, 1)$, consider $\forall n \in \mathbb{N}\ \forall x \in K$:
  - $\Box^{0}_{\ulcorner\neg} := \mathrm{id}_K =: \Diamond^{0}_{\neg\ulcorner}$;
  - $\Box^{n}_{\ulcorner\neg} := \ulcorner\neg\Box^{n-1}_{\ulcorner\neg}$, $\Diamond^{n}_{\neg\ulcorner} := \neg\ulcorner\Diamond^{n-1}_{\neg\ulcorner}$;
  - $\Box^{\mathbb{N}}_{\ulcorner\neg}x := \bigwedge_{n\in\mathbb{N}} \Box^{n}_{\ulcorner\neg}x$, $\Diamond^{\mathbb{N}}_{\neg\ulcorner}x := \bigvee_{n\in\mathbb{N}} \Diamond^{n}_{\neg\ulcorner}x$.
- They satisfy:
  - $\Box^{\mathbb{N}}_{\ulcorner\neg}(x \wedge y) = \Box^{\mathbb{N}}_{\ulcorner\neg}x \wedge \Box^{\mathbb{N}}_{\ulcorner\neg}y$, $\Diamond^{\mathbb{N}}_{\neg\ulcorner}(x \vee y) = \Diamond^{\mathbb{N}}_{\neg\ulcorner}x \vee \Diamond^{\mathbb{N}}_{\neg\ulcorner}y$;
  - $\Box^{\mathbb{N}}_{\ulcorner\neg}0 = 0 = \Diamond^{\mathbb{N}}_{\neg\ulcorner}0$, $\Box^{\mathbb{N}}_{\ulcorner\neg}1 = 1 = \Diamond^{\mathbb{N}}_{\neg\ulcorner}1$;
  - $\Diamond^{\mathbb{N}}_{\neg\ulcorner} \dashv \Box^{\mathbb{N}}_{\ulcorner\neg}$ (i.e. $\Diamond^{\mathbb{N}}_{\neg\ulcorner}x \leq y$ iff $x \leq \Box^{\mathbb{N}}_{\ulcorner\neg}y\ \forall x, y \in K$).
- So:
  1) $\Box^{\mathbb{N}}_{\ulcorner\neg}$ (resp., $\Diamond^{\mathbb{N}}_{\neg\ulcorner}$) is a necessity (resp., possibility) operator and a topological interior (resp., topological closure) operator;
  2) $(\Diamond^{\mathbb{N}}_{\neg\ulcorner}, \Box^{\mathbb{N}}_{\ulcorner\neg})$ is an adjoint modal operator couple (i.e. a 'MAO$_0$ couple' in the terminology of [Reyes–Zawadowski'93]).
- [Rauszer'70]: $\Box^{n}_{\ulcorner\neg}$, $\bigvee_{m\in\{1,\ldots,n\}} \Box^{m}_{\ulcorner\neg}$, $\Diamond^{n}_{\neg\ulcorner}$, $\bigwedge_{m\in\{1,\ldots,n\}} \Diamond^{m}_{\neg\ulcorner}$ $\forall n \in \mathbb{N}$ for any Skolem algebra (without modal interpretation).
- [López-Escobar'85]: $\Box^{1}_{\ulcorner\neg}$ (resp., $\Diamond^{1}_{\neg\ulcorner}$) is a necessity (resp., possibility) operator for any Skolem algebra.

Generalised Reyes–Zolfaghari adjoint modal operators

[Engel–Kostecki'26]:

- If $(K, \leq, \wedge, \vee, 0, 1, {}^{\circ}, {}^{\bullet})$ is a countably additively complete bi-Vakarelov algebra satisfying $x \leq x^{\circ\circ}$, $x \wedge x^{\circ} = 0$, $x^{\bullet\bullet} \leq x$, and $x \vee x^{\bullet} = 1$ (i.e. iff it is a countably additively complete bounded biprequasiintuitionistic algebra), then $\Box^{\mathbb{N}}_{\bullet\circ}$ and $\Diamond^{\mathbb{N}}_{\circ\bullet}$, constructed analogously to $\Box^{\mathbb{N}}_{\ulcorner\neg}$ and $\Diamond^{\mathbb{N}}_{\neg\ulcorner}$, satisfy all of the above properties, i.e.:
  1) $\Box^{\mathbb{N}}_{\bullet\circ}$ (resp., $\Diamond^{\mathbb{N}}_{\circ\bullet}$) is a necessity (resp., possibility) operator and a topological interior (resp., topological closure) operator;
  2) $(\Diamond^{\mathbb{N}}_{\circ\bullet}, \Box^{\mathbb{N}}_{\bullet\circ})$ is an adjoint modal operator couple.
- Every Akchurin algebra exhibits four different adjoint modal couples: $(\Diamond^{\mathbb{N}}_{\neg\ulcorner}, \Box^{\mathbb{N}}_{\ulcorner\neg})$, $(\Diamond^{\mathbb{N}}_{\circ\bullet}, \Box^{\mathbb{N}}_{\bullet\circ})$, $(\Diamond^{\mathbb{N}}_{\neg\bullet}, \Box^{\mathbb{N}}_{\bullet\neg})$, $(\Diamond^{\mathbb{N}}_{\circ\ulcorner}, \Box^{\mathbb{N}}_{\ulcorner\circ})$.
- More generally, every (n, n)-Akchurin algebra extends to an $(n, n, 4n+4, 4n+4)$-Akchurin algebra, with $2n+2$ modal pairs $(\Diamond^{1}_{\circ\bullet}, \Box^{1}_{\bullet\circ})$ and $2n+2$ Reyes–Zolfaghari adjoint modal operator couples.

Border and coborder operators: paraconsistency & paracompleteness

- [Kuratowski'22, Zarycki'27]: given a boolean algebra $(K = 2^X, \leq, \wedge, \vee, 0, 1, ^{\perp})$ of subsets of a given set X, one can introduce various topological operators (e.g.: closure, interior, exterior,...), allowing to axiomatise topology on X.
- [Zarycki'27]: In particular: the topological border operator $\partial_{\mathrm{Z}} x := x \wedge \mathrm{cl}(x^{\perp})$ is idempotent ($\partial_{\mathrm{Z}}\partial_{\mathrm{Z}} = \partial_{\mathrm{Z}}$) and satisfies the Leibniz rule: $\partial_{\mathrm{Z}}(x \wedge y) = (\partial_{\mathrm{Z}} x \wedge y) \vee (x \wedge \partial_{\mathrm{Z}} y)$ $\forall x, y \in K$.
- [Lawvere'76'86'90'91]: given a Brouwer algebra $(K, \leq, \wedge, \vee, 0, 1, \urcorner)$, a border operator $\partial_{\urcorner} x := x \wedge \urcorner x$ is idempotent, satisfies the Leibniz rule, as well as $\urcorner\urcorner x \vee \partial_{\urcorner} x = x$ $\forall x \in K$.
- [Gutiérrez Chaparro'09, Ellerman'10, Mormann'13]: a coborder operator $\mathrm{d}_{\neg} x := x \vee \neg x$ $\forall x \in K$ for Heyting algebras $(K, \leq, \wedge, \vee, 0, 1, \neg)$ satisfies properties order dual to the properties of $\partial_{\urcorner}$.
- [Kostecki'24'25]: A generalisation of a border operator to coquasiintutionistic algebras satisfies all of the above properties of $\partial_{\urcorner}$.
- [Engel–Kostecki'26]: Distributive Vakarelov algebras satisfying $x^{\bullet} \vee y^{\bullet} \geq (x \wedge y)^{\bullet}$ and $x \vee x^{\bullet} \geq y$ are the most general Vakarelov algebras s.t. the border operator $\partial_{\bullet} x := x \wedge x^{\bullet}$ satisfies (idempotency and Leibniz rule) or $x^{\bullet\bullet} \vee \partial_{\bullet} x = x$. This is equivalent with an order dual result for the border operator $\mathrm{d}_{\circ} x := x \vee x^{\circ}$ for order dual algebras.

Involutive and orthocomplemented lattices

- [Birkhoff'40]: A Vakarelov algebra $(K, \leq, \wedge, \vee, {}^{\perp})$ is called an involutive lattice iff it is both quasiminimal algebra and a coquasiminimal algebra w.r.t. to ${}^{\perp}$, i.e. iff $x = x^{\perp\perp}$ $\forall x \in K$.
- [Birkhoff–von Neumann'36]: A bounded Vakarelov algebra $(K, \leq, \wedge, \vee, 0, 1, {}^{\perp})$ is called an orthocomplemented lattice iff it satisfies $x = x^{\perp\perp}$ and $(x \wedge x^{\perp} = 0$ or $x \vee x^{\perp} = 1)$ $\forall x \in K$.
- A boolean algebra := an orthocomplemented lattice $(L, {}^{\perp})$ satisfying $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ $\forall x, y, z \in L$.
- The corresponding Vakarelov logics will be denoted by: **Inv**, **Orth**, and **Cl**, respectively.
- [Husimi'37] An orthocomplemented lattice $(L, {}^{\perp})$ is called orthomodular iff $x \vee y = ((x \vee y) \wedge y^{\perp}) \vee y$ $\forall x, y \in L$.

A generalised Kolmogorov–Glivenko theorem I

- [Kolmogorov'1925–Glivenko'29]: $\neg\neg\mathbf{Int} = \mathbf{Cl}$.
- [Rieger'49]: if $(K, \leq, \wedge, \vee, 0, 1, \neg)$ is a Heyting algebra, then $(K_{\neg\neg} := \{\neg\neg x : x \in K\}, \leq, \wedge, (\cdot \curlyvee \cdot) := \neg\neg(\cdot \vee \cdot), 0, 1, \neg)$ is a boolean algebra.
- [McKinsey–Tarski'46]: analogous algebraic result for a Brouwer algebra.
- All of the above proofs do not rely on $\rightarrow$ (resp., $\rightarrowtail$), so they apply to **PInt** (resp., **coPInt**) and (resp., co)preintuitionistic algebras.
- [Engel–Kostecki'26][2]: If $(K, \leq, \wedge, \vee, 0, 1, ^{\bullet})$ is a bounded coprequasiintuitionistic algebra, and $K_{\bullet\bullet} := \{x^{\bullet\bullet} : x \in K\}$, then $(K_{\bullet\bullet}, \leq, \curlywedge, \vee, 0, 1, ^{\bullet})$ is an orthocomplemented lattice s.t.:
  - $\leq$ (resp., $\vee$; $^{\bullet}$) is given by a restriction of $\leq$ (resp., $\vee$; $^{\bullet}$) to $K_{\bullet\bullet}$;
  - $\alpha \curlywedge \beta := (\alpha \wedge \beta)^{\bullet\bullet}\ \forall \alpha, \beta \in K_{\bullet\bullet}$;
  - $0, 1 \in K_{\bullet\bullet}$ is given by $0, 1 \in K$.
- An order dual result holds for a bounded prequasiintuitionistic algebra $(K, \leq, \wedge, \vee, 0, 1, ^{\circ})$, with an orthocomplemented lattice $(K_{\circ\circ} := \{x^{\circ\circ} : x \in K\}, \wedge, (\cdot \curlyvee \cdot) := (\cdot \vee \cdot)^{\circ\circ}, 0, 1, ^{\circ})$.

[2] It is possible that this result appeared somewhere before, but we haven't found it anywhere.

A generalised Kolmogorov–Glivenko theorem II

- If $(K, \leq, \wedge, \vee, {}^\circ)$ is a quasiminimal algebra, then $(\cdot)^{\circ\circ}$ is a closure operator, i.e. it is order monotone, $x \leq x^{\circ\circ}$, $x^{\circ\circ\circ\circ} \leq x^{\circ\circ}$.
- If $(K, \leq, \wedge, \vee, {}^\bullet)$ is a coquasiminimal algebra, then $(\cdot)^{\bullet\bullet}$ is an interior operator, i.e. it is order monotone, $x^{\bullet\bullet} \leq x$, $x^{\bullet\bullet} \leq x^{\bullet\bullet\bullet\bullet}$.
- So, the default domain of the generalised Kolmogorov–Glivenko theorem is a (resp., co)quasiminimal algebra, with the image of ${}^{\circ\circ}$ (resp., ${}^{\bullet\bullet}$) given by an involutive lattice.[3]

[3] For a quasiminimal algebra this result has appeared in [Dunn'93].

A generalised Kolmogorov–Glivenko theorem III [Engel–Kostecki'26]

The logical counterpart of this result, due to equivalence of the logic $\mathbf{K}(\vdash,\wedge,\vee,\underset{\circ}{\sim})$ (resp., $\mathbf{K}(\vdash,\wedge,\vee,\underset{\bullet}{\sim})$; $\mathbf{L}(\vdash,\wedge,\curlyvee,\underset{\circ}{\sim})$; $\mathbf{L}(\vdash,\curlywedge,\vee,\underset{\bullet}{\sim})$) with respect to the class $\mathcal{K}$ of all algebras $(K,\leq,\wedge,\vee,^\circ)$ (resp., $(K,\leq,\wedge,\vee,^\bullet)$; $(L,\leq,\wedge,\curlyvee,^\circ)$; $(K,\leq,\curlywedge,\vee,^\bullet)$), will be called a generalised Kolmorogov–Glivenko theorem:

$$\alpha_1,\ldots,\alpha_n \models_{\chi_{(\boldsymbol{L},\leq,\curlywedge,\vee,^\bullet)}} \alpha \text{ iff } (\alpha_1,\ldots,\alpha_n)^{\bullet\bullet} \models_{\chi_{(\boldsymbol{K},\leq,\wedge,\vee,^\bullet)}} \alpha \;\; \forall(K,\leq,\wedge,\vee,^\bullet)\in\mathcal{K}, \quad (2)$$

$$\beta_1,\ldots,\beta_n \models_{\chi_{(\boldsymbol{L},\leq,\wedge,\curlyvee,^\circ)}} \beta \text{ iff } (\beta_1,\ldots,\beta_n) \models_{\chi_{(\boldsymbol{K},\leq,\wedge,\vee,^\circ)}} \beta^{\circ\circ} \;\; \forall(K,\leq,\wedge,\vee,^\circ)\in\mathcal{K}, \quad (3)$$

where $\alpha,\alpha_1,\ldots,\alpha_n$ are sequents of $\mathbf{K}(\vdash,\wedge,\vee,\underset{\bullet}{\sim})$ that do not contain $\wedge$, $\beta,\beta_1,\ldots,\beta_n$ are sequents of $\mathbf{K}(\vdash,\wedge,\vee,\underset{\circ}{\sim})$ that do not contain $\vee$, χ_M denotes a valuation in the corresponding algebra M, while $\mathcal{K}$ is a class of all such algebras.

logic $\mathbf{K}(\vdash,\wedge,\vee,\sim)$	Lindenbaum algebra $(K,\leq,\wedge,\vee,^\dagger)$	internal lattice	internal logic	eqn.
QMin	quasiminimal	involutive	**Inv**	(3)
coQMin	coquasiminimal	involutive	**Inv**	(2)
$\mathbf{QMin}^{\perp,\top}$	bounded quasiminimal	bounded involutive	$\mathbf{Inv}^{\perp,\top}$	(3)
$\mathbf{coQMin}^{\perp,\top}$	bounded coquasiminimal	bounded involutive	$\mathbf{Inv}^{\perp,\top}$	(2)
$\mathbf{PQInt}^{\perp,\top}$	bounded prequasiintuitionistic	orthocomplemented	**Orth**	(3)
$\mathbf{coPQInt}^{\perp,\top}$	bounded coprequasiintuitionistic	orthocomplemented	**Orth**	(2)
PInt	preintuitionistic	boolean	**Cl**	(3)
coPInt	copreintuitionistic	boolean	**Cl**	(2)

A generalised Kolmogorov–Glivenko theorem IV [Engel–Kostecki'26]

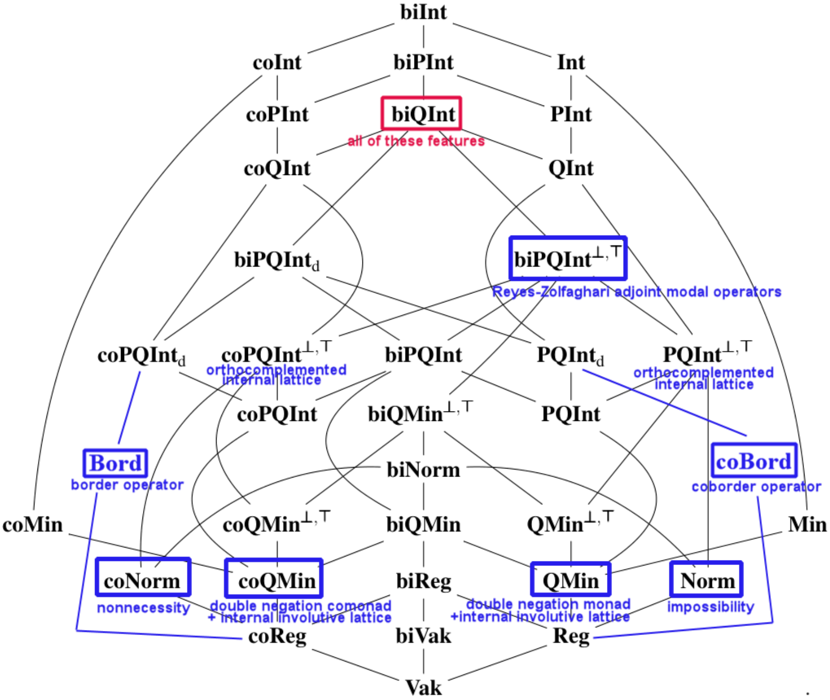
- For every bounded coprequasiintuitionistic algebra $(K, \leq, \wedge, \vee, 0, 1, ^{\bullet})$, $(K_{\bullet\bullet}, \leq, \curlywedge, \vee, 0, 1, ^{\bullet})$ is orthomodular iff

$$\forall x, y \in K_{\bullet\bullet} \setminus \{0, 1\} \text{ if } x^{\bullet} \leq y \text{ and } x \curlywedge y = 0, \text{ then } x = y.$$

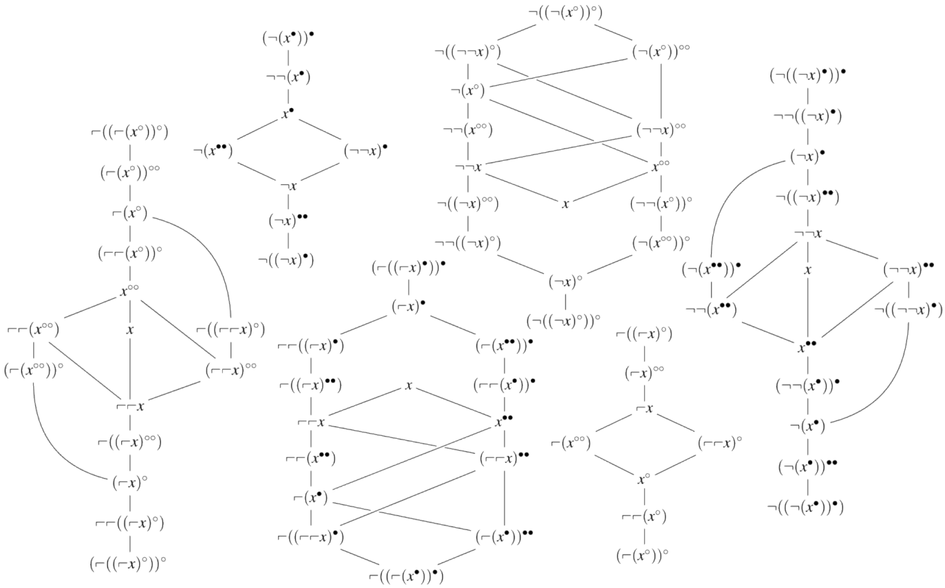
- By order duality, for every bounded prequasiintuitionistic algebra $(K, \leq, \wedge, \vee, 0, 1, ^{\circ})$, $(K_{\circ\circ}, \leq, \wedge, \curlyvee, 0, 1, ^{\circ})$ is orthomodular iff

$$\forall x, y \in K_{\circ\circ} \setminus \{0, 1\} \text{ if } x \leq y^{\circ} \text{ and } x \curlyvee y = 1, \text{ then } x = y.$$

- For every distributive coprequasiintuitionistic algebra $(K, \leq, \wedge, \vee, ^{\bullet})$, the formula $\partial_{\bullet} x \vee x^{\bullet\bullet}$ $\forall x \in K$ reads: every element of K is an (extensional) alternative of its paraconsistent border and its double negation interior.
- Any Akchurin algebra has:
  - two (a priori different) 'internal' orthocomplemented lattices;
  - two (isomorphic) 'internal' boolean lattices;
  - two (a priori different) border operators;
  - two (a priori different) coborder operators.


biInt
coInt
biPInt
Int
coPInt
biQInt
PInt
all of these features
coQInt
QInt
biPQInt$_{\mathrm{d}}$
biPQInt$^{\perp,\top}$
Reyes-Zolfaghari adjoint modal operators
coPQInt$_{\mathrm{d}}$
coPQInt$^{\perp,\top}$
orthocomplemented
internal lattice
biPQInt
PQInt$_{\mathrm{d}}$
PQInt$^{\perp,\top}$
orthocomplemented
internal lattice
coPQInt
biQMin$^{\perp,\top}$
PQInt
Bord
border operator
biNorm
coBord
coborder operator
coMin
coQMin$^{\perp,\top}$
biQMin
QMin$^{\perp,\top}$
Min
coNorm
coQMin
biReg
QMin
Norm
nonnecessity
double negation comonad
+ internal involutive lattice
double negation monad
+internal involutive lattice
impossibility
coReg
biVak
Reg
Vak

Hasse diagrams for negations in an Akchurin algebra [Engel–Kostecki'26]



Plan:

I. [E–K'26] **Structural investigations in Vakarelov logics/algebras**:

1) biquasiintuitionistic and Akchurin logics/algebras;
2) generalised Reyes–Zolfaghari adjoint modal couples;
3) generalised Zarycki–Lawvere paraconsistent border operator;
4) generalised Kolmogorov–Glivenko translation.

II. [E–K'26] **Structure of the lattice $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ of clopen spectral subpresheaves over an orthocomplemented lattice L**:

1) characterisation of full paraconsistency of Brouwer and coquasiintuitionistic algebra reducts of $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$;
2) reconstruction of L by the generalised Kolmogorov–Glivenko translation;
3) adjunctions between spectral and cospectral lattices, and a multitude of induced negations;
4) $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ as a $(3,3)$-Akchurin algebra with 16 generalised Reyes–Zolfaghari adjoint modal couples.

III. [K'24'25'26] **Some applications to**:

1) causal structure of lorentzian spacetimes;
2) lattices of *- and W*- factor subalgebras;
3) algebraic quantum field theory.

Stone spectrum and Stone duality

- $B :=$ a boolean algebra.
- $\mathrm{sp}_{\mathrm{S}}(B) :=$ the set of all boolean lattice homomorphisms $B \to \{0, 1\}$.
- [Stone'34'37] duality: $\varkappa_B : B \to \mathrm{clop}(\mathrm{sp}_{\mathrm{S}}(B))$ is a boolean lattice isomorphism, where:
  $\mathrm{clop}(\cdot) :=$ the boolean algebra of closed-and-open sets, with:
  $\cdot \wedge \cdot := \mathrm{int}(\cdot \cap \cdot)$,
  $\cdot \vee \cdot := \mathrm{cl}(\cdot \cup \cdot)$,
  $x^{\perp} := \mathrm{sp}_{\mathrm{S}}(B) \setminus x$,
  'closed-and-open' w.r.t. a totally disconnected, compact, Hausdorff topology generated by the basis $\{\{x \in \mathrm{sp}_{\mathrm{S}}(B) \,\mathbf{:}\, x(y) = 1\} \,\mathbf{:}\, y \in B\}$.

Spectral presheaf Σ^L [Hamilton–Isham–Butterfield'00, Cannon'13, Cannon–Döring'18]

- $L :=$ an orthomodular lattice.
- $\mathcal{V}_{\mathrm{c}}(L) :=$ a poset of complete boolean subalgebras of L, ordered by inclusion, and excluding $\{0, 1\}$.
- [Cannon'13, Cannon–Döring'18][4]: the spectral presheaf := a contravariant functor $\Sigma^L : \mathcal{V}_{\mathrm{c}}(L) \to$ **Set**, acting by $B \to \mathrm{sp}_{\mathrm{S}}(B)$ on objects and by restriction maps $(B \subseteq D) \mapsto (\mathrm{sp}_{\mathrm{S}}(D) \to \mathrm{sp}_{\mathrm{S}}(D)|_B)$ on morphisms.
- [Cannon'13, Cannon–Döring'18]: any two orthomodular lattices are isomorphic (in the category of orthomodular lattices and orthomodular lattice homomorphisms) iff their corresponding spectral presheaves are isomorphic (in the category of Stone space-valued presheaves with morphisms given by pairs of a homomorphism of underlying orthomodular lattice and a natural transformation between the corresponding presheaves).

---

[4] [Hamilton–Isham–Butterfield'00]: earlier variant, $\Sigma^{\mathcal{N}}$, with $\mathcal{V}_{\mathrm{c}}(L)$ replaced by the poset $\mathcal{V}(\mathcal{N})$ of commutative von Neumann subalgebras of a given von Neumann algebra $\mathcal{N}$, and with the Stone spectrum sp_{S} replaced by the Gel'fand spectrum (i.e. the set of all nonzero *-algebra homomorphisms from a commutative von Neumann algebra to $\mathbb{C}$). This formulation is equivalent the choice $L = \mathrm{Proj}(\mathcal{N}) = \{x \in \mathcal{N} : x = x^2 = x^*\}$ in the setting of Σ^L. Such L is complete and orthomodular.

$\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ as a complete Skolem algebra I [Döring–Isham'08, Döring'12, Cannon'13, Cannon–Döring'18]

- $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L) :=$ the set of all subpresheaves F of Σ^L s.t. $F(B)$ is closed-and-open $\forall B \in \mathcal{V}_{\mathrm{c}}(L)$.
- $\leq$ is given by the subpresheaf relation, i.e. $F \leq G$ iff

$F(B) \subseteq G(B)$ and
$$\begin{tikzcd} F(B) \arrow[r, "\subseteq"] \arrow[d, "\cdot|_B"] & G(B) \arrow[d, "\cdot|_B"] \\ F(D) \arrow[r, "\subseteq"] & G(D) \end{tikzcd}$$
commutes $\forall B, D \in \mathcal{V}_{\mathrm{c}}(L)$.

- $1 := \Sigma^L$.
- $0 := \varnothing$ (the empty presheaf).
- $(\bigwedge_i F^i)(B) := \mathrm{int}(\bigcap_i F^i(B))$.
- $(\bigvee_i F^i)(B) := \mathrm{cl}(\bigcup_i F^i(B))$.
- [Cannon'13, Cannon–Döring'18][5]: $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1)$ is a complete Skolem algebra.

[5] [Döring–Isham'08, Döring'12]: Earlier variant for $\mathcal{V}(\mathcal{N})$ and $\Sigma^{\mathcal{N}}$.

$\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ as a complete Skolem algebra II [Engel–Kostecki'26]

- From now on, let L be orthocomplemented lattice.
- The results and constructions of [Cannon'13, Cannon–Döring'18] hold also at this level of generality, except of characterisation of L (up to isomorphism) by their spectral presheaves. In particular, $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1)$ is a complete Skolem algebra.
- $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1, \lrcorner)$ is fully paraconsistent, i.e.

$$\partial_{\lrcorner} S := S \wedge \lrcorner S \neq 0 \;\; \forall S \in \mathrm{Sub}_{\mathrm{clop}}(\Sigma^L) \setminus \{0, 1\},$$

iff $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1, \neg)$ is fully paracomplete, i.e.

$$\mathrm{d}_{\neg} S := S \vee \neg S \neq 1 \;\; \forall S \in \mathrm{Sub}_{\mathrm{clop}}(\Sigma^L) \setminus \{0, 1\},$$

iff $\lrcorner S > \neg S \; \forall S \in \mathrm{Sub}_{\mathrm{clop}}(\Sigma^L) \setminus \{0, 1\}$
iff

$$\Box^{\mathbb{N}}_{\lrcorner\neg} S = 0 \;\text{ and }\; \Diamond^{\mathbb{N}}_{\neg\lrcorner} S = 1 \;\; \forall S \in \mathrm{Sub}_{\mathrm{clop}}(\Sigma^L) \setminus \{0, 1\}$$

iff ($L \neq \{0, x, x^{\perp}, 1\}$ and $\mathcal{V}_{\mathrm{c}}(L)$ is connected).

Outer daseinisation [Döring–Isham'08, Cannon'13, Cannon–Döring'18]

- Given $x \in L$, an outer daseinisation := a functor $\delta^{\wedge}(x) : (\mathcal{V}_{\mathrm{c}}(L))^{\mathrm{op}} \to \mathbf{Set}$ s.t.:

$$(\delta^{\wedge}(x))(B) := \underbrace{\left\{ s \in \mathrm{sp}_{\mathrm{S}}(B) \,:\, s\left(\underbrace{\bigwedge\{y \in B \,:\, y \geq x\}}_{\text{best approx. of } x \text{ in } B \text{ from above}} \right) = 1 \right\}}_{\text{elements of } \mathrm{sp}_{\mathrm{S}}(B) \text{ for which the best approx. of } x \text{ holds}} \forall B \in \mathrm{Ob}(\mathcal{V}_{\mathrm{c}}(L))$$

  and it is mapping the morphisms of $\mathcal{V}_{\mathrm{c}}(L)$ to the restriction maps of Stone spectra, i.e. $\delta^{\wedge}(x) : (B \subseteq D) \mapsto (\mathrm{sp}_{\mathrm{S}}(D) \to \mathrm{sp}_{\mathrm{S}}(D)|_B)$.
- This construction extends to a functor $\delta^{\wedge} : L \to \mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$.
- $\delta^{\wedge}$ is order monotone, injective, and preserves 0, 1, and all joins that exist in L.
- if L has all joins, then there is a right adjoint map $\varepsilon^{\wedge}$, i.e. $\delta^{\wedge} \dashv \varepsilon^{\wedge}$, where

$$\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L) \ni S \mapsto \varepsilon^{\wedge}(S) := \bigvee\{a \in L \,:\, \delta^{\wedge}(a) \leq S\} \in L$$

  is order monotone, surjective, and preserves 0, 1, and all meets.

Coquasiintuitionistic structure of $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ [Eva'15, Engel–Kostecki'26]

- Let L have all joins.
- [Eva'15] (under assumption that $L = \mathrm{Proj}(\mathfrak{B}(\mathcal{H}))$):
  1) $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L) \ni S \mapsto S^\bullet := \delta^\wedge \circ {}^\perp \circ \varepsilon^\wedge(S) \in \mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$.
  2) $x \wedge x^\bullet \geq 0$, so this is a paraconsistent negation.
  3) *Claim (without a proof):* $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1, {}^\bullet)$ is a model of a relevance logic **DK**.
- [Engel–Kostecki'26]:
  1. $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1, {}^\bullet)$ is a coquasiintuitionistic algebra.
  2. No-go theorem (with a proof): $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1, {}^\bullet)$ cannot be a nontrivial model of *any* relevance logic.
  3. The generalised Kolmorogov–Glivenko theorem reconstructs the underlying lattice: $(L, \leq, \wedge, \vee, 0, 1, {}^\perp) \cong ((\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L))_{\bullet\bullet}, \leq, \curlywedge, \vee, 0, 1, {}^\bullet)$.
  4. $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1, {}^\bullet)$ is fully paraconsistent iff $L \neq \{0, x, x^\perp, 1\}$.

Inner daseinisation, take one [Döring–Isham'08'11, Engel–Kostecki'26]

- [Döring–Isham'08]: Let $x \in L$, and consider an inner daseinisation functor $\delta^{\vee}(x) : (\mathcal{V}_{\mathrm{c}}(L))^{\mathrm{op}} \to \mathbf{Set}$ acting by

$$(\delta^{\vee}(x))(B) := \underbrace{\left\{ s \in \mathrm{sp}_{\mathrm{S}}(B) \,:\, s\left(\underbrace{\bigvee\{y \in B \,:\, y \leq x\}}_{\text{best approx. of } x \text{ in } B \text{ from below}} \right) = 1 \right\}}_{\text{elements of } \mathrm{sp}_{\mathrm{S}}(B) \text{ for which the best approx. of } x \text{ holds}} \forall B \in \mathrm{Ob}(\mathcal{V}_{\mathrm{c}}(L))$$

  and by $(B \subseteq D) \mapsto (\mathrm{sp}_{\mathrm{S}}(D) \to \mathrm{sp}_{\mathrm{S}}(D)|_{B})$ on morphisms of $\mathcal{V}_{\mathrm{c}}(L)$.
- [Döring–Isham'11]: Inner daseinisation defines a functor $\delta^{\vee} \circ^{\perp} : L \to \mathrm{Sub}_{\mathrm{clop}}(\Sigma^{L})$ acting by $\delta^{\vee}(x^{\perp})$ $\forall x \in L$.
- [Engel–Kostecki'26]: The image of $\delta^{\vee}$ (without $\circ^{\perp}$) is inside $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^{L})$ iff ($L = \{0, x, x^{\perp}, 1\}$ or $\mathcal{V}_{\mathrm{c}}(L)$ is completely disjoint).
- This result forbids consideration of a quasiintutionistic negation $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^{L}) \ni S \mapsto S^{\circ} := \delta^{\vee} \circ {}^{\perp} \circ \varepsilon^{\vee}(S) \in \mathrm{Sub}_{\mathrm{clop}}(\Sigma^{L})$ that would be defined by an adjunction $\delta^{\vee} \dashv \varepsilon^{\vee}$.
- However, this is not the end of the story...

Spectral and cospectral lattices [Marsden'10, Engel–Kostecki'26]

- A spectral map is an order monotone map $\sigma : (\mathcal{V}_{\mathrm{c}}(L))^{\mathrm{op}} \ni V \mapsto \sigma_V \in L$ s.t. $\sigma_V \in V \; \forall V \in \mathcal{V}_{\mathrm{c}}(L)$.
- A cospectral map is an order monotone map $\vartheta : \mathcal{V}_{\mathrm{c}}(L) \ni V \mapsto \vartheta_V \in L$ s.t. $\vartheta_V \in V \; \forall V \in \mathcal{V}_{\mathrm{c}}(L)$.
- [Marsden'10] (resp., [Engel–Kostecki'26]): A spectral (resp., cospectral) lattice := $\mathfrak{S}(L)$ (resp., $\mathfrak{S}^{\mathrm{op}}(L)$) := the set of all (resp., co)spectral maps.
- [Marsden'10]: $\varkappa : \mathfrak{S}(L) \to \mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$, given by $(\varkappa(\sigma))_V := \varkappa_V(\sigma_V)$ $\forall \sigma \in \mathfrak{S}(L) \; \forall V \in \mathcal{V}_{\mathrm{c}}(L)$, is a complete lattice isomorphism.
- Hence $\mathfrak{S}(L)$ is a complete Skolem algebra.
- $\Upsilon : \mathfrak{S}(L) \to \mathfrak{S}^{\mathrm{op}}(L)$ is defined by $(\Upsilon(\sigma))_V := {\sigma_V}^{\perp} \; \forall \sigma \in \mathfrak{S}(L) \; \forall V \in \mathcal{V}_{\mathrm{c}}(L)$.
- $\Upsilon^{-1} : \mathfrak{S}^{\mathrm{op}}(L) \to \mathfrak{S}(L)$ is defined by $(\Upsilon^{-1}(\vartheta))_V := {\vartheta_V}^{\perp} \; \forall \vartheta \in \mathfrak{S}^{\mathrm{op}}(L)$ $\forall V \in \mathcal{V}_{\mathrm{c}}(L)$.
- Υ and Υ^{-1} form an order antimonotone complete lattice isomorphism.
- Hence $\mathfrak{S}^{\mathrm{op}}(L)$ is a complete Skolem algebra.
- Heyting (resp., Brouwer) negation on $\mathfrak{S}^{\mathrm{op}}(L)$ will be denoted by $\urcorner$ (resp., $\llcorner$).

Inner daseinisation, take two [Engel–Kostecki'26]

- $\bar{\delta}^{\vee} : L \to \mathfrak{S}^{\mathrm{op}}(L)$, where $(\bar{\delta}^{\vee}(a))_V := \bigvee\{b \in V \,\mathbf{:}\, b \leq a\}\ \forall a \in L\ \forall V \in \mathcal{V}_{\mathrm{c}}(L)$, is an injective bounded meet semilattice homomorphism, preserving all meets that exist in L.
- If L has all meets, then $\bar{\varepsilon}^{\vee} : \mathfrak{S}^{\mathrm{op}}(L) \ni \vartheta \mapsto \bar{\varepsilon}^{\vee}(\vartheta) := \bigwedge\{a \in L \,\mathbf{:}\, \vartheta \leq \bar{\delta}^{\vee}(a)\} \in L$ is a surjective bounded join semilattice homomorphism that preserves all joins.
- $\bar{\delta}^{\wedge} : L \to \mathfrak{S}(L)$ and $\bar{\varepsilon}^{\wedge} : \mathfrak{S}(L) \to L$ are defined analogously.
- If L has all meets, then $(\cdot)^{\square} := \bar{\delta}^{\vee} \circ {}^{\perp} \circ \bar{\varepsilon}^{\vee} : \mathfrak{S}^{\mathrm{op}}(L) \to \mathfrak{S}^{\mathrm{op}}(L)$ is a quasiintuitionistic negation.
- If L is complete, then:
  - $\bar{\delta}^{\wedge} \circ \bar{\varepsilon}^{\vee} : \mathfrak{S}^{\mathrm{op}}(L) \rightleftarrows \mathfrak{S}(L) : \bar{\delta}^{\vee} \circ \bar{\varepsilon}^{\wedge}$ form an adjunction $\bar{\delta}^{\wedge} \circ \bar{\varepsilon}^{\vee} \dashv \bar{\delta}^{\vee} \circ \bar{\varepsilon}^{\wedge}$;
  - $(\cdot)^{\star} := \bar{\delta}^{\wedge} \circ \bar{\varepsilon}^{\vee} \circ \llcorner \circ \bar{\delta}^{\vee} \circ \bar{\varepsilon}^{\wedge} : \mathfrak{S}(L) \to \mathfrak{S}(L)$ is a coquasiintuitionistic negation;
  - $(\cdot)^{\bowtie} := \bar{\delta}^{\vee} \circ \bar{\varepsilon}^{\wedge} \circ \lrcorner \circ \bar{\delta}^{\wedge} \circ \bar{\varepsilon}^{\vee} : \mathfrak{S}^{\mathrm{op}}(L) \to \mathfrak{S}^{\mathrm{op}}(L)$ is a quasiintuitionistic negation.

Some more adjunctions... [Engel–Kostecki'26]

- $\mathcal{K} : \mathfrak{S}(L) \to \mathfrak{S}^{\mathrm{op}}(L)$ is defined by $(\mathcal{K}(\sigma))_V := \bigwedge_{Y \in \uparrow V} \bar{\delta}^{\vee}_V(\sigma_Y)$ $\forall \sigma \in \mathfrak{S}(L)$ $\forall V \in \mathcal{V}_{\mathrm{c}}(L)$;
- $\mathcal{G}_{\mathrm{op}} := \Upsilon^{-1} \circ \mathcal{K} \circ \Upsilon^{-1}$ satisfies $(\mathcal{G}_{\mathrm{op}}(\vartheta))_V = \bigvee_{Y \in \uparrow V} \bar{\delta}^{\wedge}_V(\vartheta_Y)$;
- $\mathcal{G} : \mathfrak{S}(L) \to \mathfrak{S}^{\mathrm{op}}(L)$ is defined by $(\mathcal{G}(\sigma))_V := \bigvee_{Y \in \downarrow V} \sigma_Y$ $\forall \sigma \in \mathfrak{S}(L)$ $\forall V \in \mathcal{V}_{\mathrm{c}}(L)$;
- $\mathcal{K}_{\mathrm{op}} := \Upsilon^{-1} \circ \mathcal{G} \circ \Upsilon^{-1}$ satisfies $(\mathcal{K}_{\mathrm{op}}(\vartheta))_V = \bigwedge_{Y \in \downarrow V} \vartheta_Y$.
- $\mathcal{G}_{\mathrm{op}} \dashv \mathcal{K}$ and $\mathcal{G} \dashv \mathcal{K}_{\mathrm{op}}$.
- The following diagram commutes:

$$
\begin{tikzcd}
\mathfrak{S}^{\mathrm{op}}(L) \arrow[rr, "\llcorner"] \arrow[dr, "\Upsilon^{-1}"] \arrow[ddr, bend right, "\mathrm{id}_{\mathfrak{S}^{\mathrm{op}}(L)}"'] & & \mathfrak{S}^{\mathrm{op}}(L) \arrow[ddd, bend right, "\Upsilon^{-1}"'] & & \mathfrak{S}^{\mathrm{op}}(L) \arrow[ll, "\lrcorner"'] \arrow[dl, "\Upsilon^{-1}"'] \arrow[ddl, bend left, "\mathrm{id}_{\mathfrak{S}^{\mathrm{op}}(L)}"] \\
 & \mathfrak{S}(L) \arrow[ur, "\mathcal{G}"] \arrow[d, bend left, "\Upsilon"] & & \mathfrak{S}(L) \arrow[ul, "\mathcal{K}"'] \arrow[d, bend left, "\Upsilon"] & \\
 & \mathfrak{S}^{\mathrm{op}}(L) \arrow[u, bend left, "\Upsilon^{-1}"] \arrow[dr, "\mathcal{K}_{\mathrm{op}}"] & & \mathfrak{S}^{\mathrm{op}}(L) \arrow[u, bend left, "\Upsilon^{-1}"] \arrow[dl, "\mathcal{G}_{\mathrm{op}}"'] & \\
\mathfrak{S}(L) \arrow[rr, "\neg"'] \arrow[ur, "\Upsilon"] \arrow[uur, bend left, "\mathrm{id}_{\mathfrak{S}(L)}"] & & \mathfrak{S}(L) \arrow[uuu, bend right, "\Upsilon"'] & & \mathfrak{S}(L). \arrow[ll, "\ulcorner"] \arrow[ul, "\Upsilon"'] \arrow[uul, bend right, "\mathrm{id}_{\mathfrak{S}(L)}"']
\end{tikzcd}
$$

...and some more negations (which are distinct) [Engel–Kostecki'26]

- $(\cdot)^{\triangledown} := \mathcal{K}_{\mathrm{op}} \circ \lrcorner \circ \mathcal{G} : \mathfrak{S}(L) \to \mathfrak{S}(L)$ is a quasiintuitionistic negation.
- $(\cdot)^{\blacktriangledown} := \mathcal{G}_{\mathrm{op}} \circ \llcorner \circ \mathcal{K} : \mathfrak{S}(L) \to \mathfrak{S}(L)$ is a coquasiintuitionistic negation.
- If L has all meets, then $(\cdot)^{\circ} := \mathcal{K}_{\mathrm{op}} \circ {}^{\square} \circ \mathcal{G} = \mathcal{K}_{\mathrm{op}} \circ \bar{\delta}^{\vee} \circ {}^{\perp} \circ \bar{\varepsilon}^{\vee} \circ \mathcal{G} : \mathfrak{S}(L) \to \mathfrak{S}(L)$ is a quasiintuitionistic negation.
- If L is complete, then $(\cdot)^{\star} := \mathcal{K}_{\mathrm{op}} \circ {}^{\bowtie} \circ \mathcal{G} = \mathcal{K}_{\mathrm{op}} \circ \bar{\delta}^{\vee} \circ \bar{\varepsilon}^{\wedge} \circ \lrcorner \circ \bar{\delta}^{\wedge} \circ \bar{\varepsilon}^{\vee} \circ \mathcal{G} : \mathfrak{S}(L) \to \mathfrak{S}(L)$ is a quasiintuitionistic negation.

- $(\cdot)^{\triangledown} = \neg$ iff $\mathcal{V}_{\mathrm{c}}(L)$ is completely disjoint.
- $(\cdot)^{\blacktriangledown} = \ulcorner$ iff $\mathcal{V}_{\mathrm{c}}(L)$ is completely disjoint.
- If L has all meets, then $(\cdot)^{\circ} = \neg$ iff $L \neq \{0, x, x^{\perp}, 1\}$.
- If L has all meets, then $(\cdot)^{\circ} = (\cdot)^{\triangledown}$ iff $\forall a, b \in L \setminus \{0, 1\}\ \exists U \in \mathcal{V}_{\mathrm{c}}(L) :\ a, b \in U$.
- If L has all joins, then $(\cdot)^{\bullet} = \ulcorner$ iff L is a boolean algebra.
- If L has all joins, then $(\cdot)^{\bullet} = (\cdot)^{\blacktriangledown}$ iff $L = \{0, x, x^{\perp}, 1\}$.
- If $L \neq \{0, x, x^{\perp}, 1\}$ is complete and orthomodular, then $(\cdot)^{\bigstar} = (\cdot)^{\times}$ and $(\cdot)^{\star} = (\cdot)^{\div}$, where $x^{\times} := \begin{cases} 1 & : x \neq 1 \\ 0 & : \text{otherwise} \end{cases}$ and $x^{\div} := \begin{cases} 0 & : x \neq 0 \\ 1 & : \text{otherwise.} \end{cases}$

Akchurin algebra structure of $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ [Engel–Kostecki'26]

- By the order monotone complete lattice isomorphism $\varkappa \circ (\cdot) \circ \varkappa^{-1}$, all negations on $\mathfrak{S}(L)$ are transported to $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$.
- $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1, {}^{\triangledown}, {}^{\blacktriangledown})$ is an Akchurin algebra.
- if L has all meets, then $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1, {}^{\triangledown}, {}^{\blacktriangledown}, {}^{\circ})$ is a $(2,1)$-Akchurin algebra.
- if L has all joins, then $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1, {}^{\triangledown}, {}^{\blacktriangledown}, {}^{\bullet})$ is a $(1,2)$-Akchurin algebra.
- if L is complete, then $(\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L), \leq, \wedge, \vee, 0, 1, {}^{\triangledown}, {}^{\blacktriangledown}, {}^{\circ}, {}^{\bullet}, {}^{\star}, {}^{\bigstar})$ is a $(3,3)$-Akchurin algebra.
- So, if L is complete, then $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ is a $(3,3,32,32)$-Akchurin algebra equipped with 16 modal operator pairs $(\Diamond^{1}_{\dagger\tilde{\dagger}}, \Box^{1}_{\tilde{\dagger}\dagger})$ and 16 Reyes–Zolfaghari adjoint modal couples $(\Diamond^{\mathbb{N}}_{\dagger\tilde{\dagger}}, \Box^{\mathbb{N}}_{\tilde{\dagger}\dagger})$, with $\dagger \in \{\neg, {}^{\triangledown}, {}^{\circ}, {}^{\star}\}$ and $\tilde{\dagger} \in \{\lrcorner, {}^{\blacktriangledown}, {}^{\bullet}, {}^{\bigstar}\}$.

Plan:

I. [E–K'26] **Structural investigations in Vakarelov logics/algebras**:

1) biquasiintuitionistic and Akchurin logics/algebras;
2) generalised Reyes–Zolfaghari adjoint modal couples;
3) generalised Zarycki–Lawvere paraconsistent border operator;
4) generalised Kolmogorov–Glivenko translation.

II. [E–K'26] **Structure of the lattice $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ of clopen spectral subpresheaves over an orthocomplemented lattice** L:

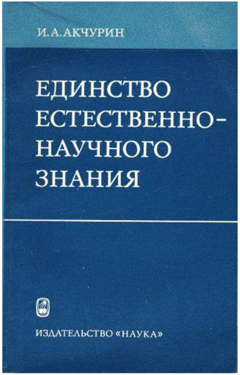
1) characterisation of full paraconsistency of Brouwer and coquasiintuitionistic algebra reducts of $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$;
2) reconstruction of L by the generalised Kolmogorov–Glivenko translation;
3) adjunctions between spectral and cospectral lattices, and a multitude of induced negations;
4) $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ as a $(3,3)$-Akchurin algebra with 16 generalised Reyes–Zolfaghari adjoint modal couples.

III. [K'24'25'26] **Some applications to**:

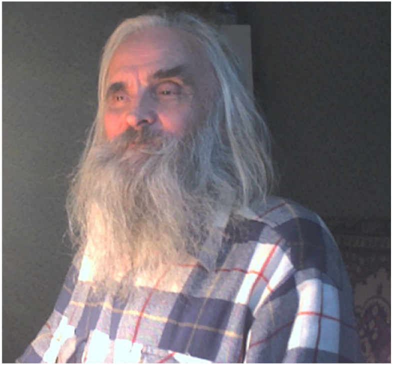
1) causal structure of lorentzian spacetimes;
2) lattices of *- and W*- factor subalgebras;
3) algebraic quantum field theory.

Akchurin's postulate

A postulate to «analyse possible connections of the state vector of a quantum system in a Hilbert space with the structures of the new spaces of Grothendieck», with a suitable topos constructed «as a category of all sheaves of wave functions»
— Igor' A. Akchurin, 1974, *Unity of the natural-scientific knowledge*, Nauka, Moskva.



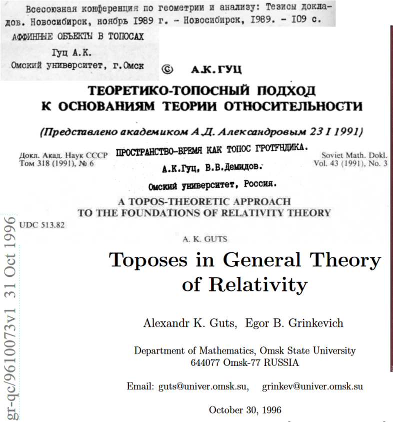
1974


Igor' A. Akchurin (1930–2005)
source of the photo: iphras.ru/uplfile/natsc/foto/akchurin_big.jpg

+ the idea to use order-theoretic axiomatisations of special/general relativity by [Aleksandrov'50, Aleksandrov–Ovchinnikova'53, Zeeman'64]/[Kronheimer–Penrose'67, Busemann'67, Pimenov'68] to construct a base category for a topos-theoretic reformulation of special/general relativity.

First implementations of Akchurin's postulates

- Aleksandr K. Guc's research programme of topos theoretic SR and GR:
  [Guc'88–'95]: toposes based on order axiomatisation of SR and GR
  [Guc'95–'24]: GR in topos models of synthetic differential geometry



Всесоюзная конференция по геометрии и анализу: Тезисы докладов. Новосибирск, ноябрь 1989 г. – Новосибирск, 1989. – 109 с.

АФФИННЫЕ ОБЪЕКТЫ В ТОПОСАХ

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ТЕОРЕТИКО-ТОПОСНЫЙ ПОДХОД К ОСНОВАНИЯМ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ

(Представлено академиком А.Д. Александровым 23 I 1991)

Докл. Акад. Наук СССР Том 318 (1991), № 6

ПРОСТРАНСТВО-ВРЕМЯ КАК ТОПОС ГРОТЕНДИКА.

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Soviet Math. Dokl. Vol. 43 (1991), No. 3

A TOPOS-THEORETIC APPROACH TO THE FOUNDATIONS OF RELATIVITY THEORY

UDC 513.82

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gr-qc/9610073v1 31 Oct 1996

Toposes in General Theory of Relativity

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October 30, 1996

Elements of time theory, 2004
(Изд. Наследие. Диалог-Сибирь, Омск)

Aleksandr K. Guc (1947–2026)
source of photo: chronology.org.ru/images/7/72/AK_Guts.jpg

- [Vasyukov'89'05]: soundness and completeness of several systems of quantum logic in a topos $\mathbf{Set}^L$ of presheaves over an orthomodular lattice L.
- [Adelman–Corbett'95'01,Corbett–Durt'09,Corbett'12'19]: sheaf of germs of convex functions on normalised trace class operators on a separable Hilbert space.

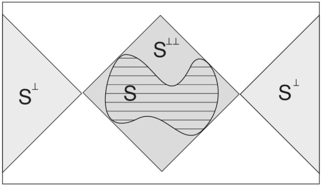
Three special cases of an orthocomplemented L in theoretical physics

- L is a complete orthomodular lattice of projective units in an archimedean order unit space [Alfsen–Shultz'76]. Special cases:
  - a lattice of idempotents in a JBW-algebra [Topping'65, Sarymsakov–Ayupov–Hadzhiev–Chilin'83];
  - a lattice of projections in a von Neumann algebra [≤Loomis'55];
  - a lattice of projection operators on a Hilbert space [von Neumann'32, Birkhoff–von Neumann'36, Husimi'37].
- L is an orthocomplemented (not necessarily orthomodular) lattice of factor von Neumann subalgebras of a factor von Neumann algebra $\mathcal{N}$, with $^{\perp}$ given by a commutant $\mathcal{A}^{\perp} := \{x \in \mathcal{N} : xy = yx \ \forall y \in \mathcal{A}\}$, $\mathcal{A}_1 \wedge \mathcal{A}_2 := \mathcal{A}_1 \cap \mathcal{A}_2$, $\mathcal{A}_1 \vee \mathcal{A}_2 := (\mathcal{A}_1 \cup \mathcal{A}_2)^{\perp\perp}$, and $\mathcal{A}$ being a factor iff $\mathcal{A} \vee \mathcal{A}^{\perp} = \mathcal{N}$ [von Neumann'29, Murray–von Neumann'36].
- L is a complete orthocomplemented lattice of subsets of a lorentzian space-time manifold (M, g) that are the image of a closure operator $(\cdot)^{\perp\perp}$ on M, where $S^{\perp} := \{$all $x \in M$ that are not connected with S by a curve of a special type$\}$. Special cases:
  - orthomodular lattice for time-like curves [Cegła–Jadczyk'77'79, Cegła–Florek'79'81'05'06, Cegła'89, Casini'02'03, Nobili'06, Cegła–Florek–Jancewicz'17];
  - nonorthomodular lattice for time-like and null curves [Casini'02'03].

Causal logics (I)

[Cegła–Jadczyk'77'79, Cegła–Florek'79'81'05'06, Casini'02'03, Cegła–Florek–Jancewicz'17]

- Let (M, g) := arbitrary (≥ 2)-dimensional lorentzian space-time.
- For $S \subseteq M$, let $S^{\perp} := \{$all $x \in M$ not connected with S by a time-like curve$\}$.



source: H. Casini, 2002, Class. Quant. Grav. **19**, 6389-6404.

- $(\cdot)^{\perp\perp}$ is a closure operator on the complete distributive lattice $(2^M, \subseteq, \cap, \cup, \varnothing, M)$.
- The set $L_{(M,g)}$ of subsets $S \subseteq M$, s.t. $S = S^{\perp\perp}$, equipped with
  - $S_1 \leq S_2 := S_1 \subseteq S_2$,
  - $S_1 \wedge S_2 := S_1 \cap S_2$,
  - $S_1 \vee S_2 := (S_1 \cup S_2)^{\perp\perp}$,

  is a complete orthomodular lattice.
- Boolean subalgebras of $L_{(M,g)}$ = sets generated by $(\cdot)^{\perp\perp}$ from the subsets of achronal surfaces of (M, g).
- [Casini'02]: If (M, g) is simply connected, then the centre of $(L_{(M,g)}, {}^{\perp})$ is nontrivial iff (M, g) admits closed time-like curves.

Causal logics (II)

- [Casini'02'03]: If $S^{\perp} := \{$all $x \in M$ not connected with S by a time-like or null-like curve$\}$, then $L_{(M,g)}$ is orthocomplemented but not orthomodular.
- [Casini'02'03]: Causal lattices defined by some discretised space-time models are also orthocomplemented but not orthomodular.
- [Nobili'06]: «(...) it is difficult to define unambiguously (...) the boundaries of causal completions».
- [Cegła'89]: $L_{(M,g)}$ does not satisfy so-called "covering property"[6], that is always satisfied by the orthomodular lattice of projections $L_{\mathrm{Proj}(\mathcal{N})}$ in any W*-algebra $\mathcal{N}$, so $L_{(M,g)}$ cannot be faithfully represented by $L_{\mathrm{Proj}(\mathcal{N})}$.
- [Kostecki'26]: From the generalised Kolmogorov–Glivenko theorem it follows that the construction of the causal logics holds in more general setting, i.e.: if $\mathrm{Sub}(M) \subseteq 2^M$ and $(\mathrm{Sub}(M), \subseteq, \cap, \cup, \varnothing, M, {}^{\perp})$ is a bounded coprequasiintuitionistic algebra, then $((\mathrm{Sub}(M))_{\perp\perp}, \subseteq, \cap, (\cdot \curlyvee \cdot) := (\cdot \vee \cdot)^{\perp\perp}, \varnothing, M, {}^{\perp})$ is an orthocomplemented lattice. Furthermore, it is also orthomodular iff

$$\forall x, y \in (\mathrm{Sub}(M))_{\perp\perp} \setminus \{\varnothing, M\} \text{ if } x \leq y^{\perp} \text{ and } x \curlyvee y = 1, \text{ then } x = y.$$

[6] An orthomodular lattice $(L, {}^{\perp})$ has the covering property iff

$$\forall x, y \in L \text{ if } (y \text{ is an atom}, y \vee y^{\perp} \neq 1) \text{ then } x \wedge (x^{\perp} \vee y) \text{ is an atom},$$

where $z \in L$ is an atom iff $0 < z$ and there exists no $w \in L$ s.t. $0 < w < z$.

Orthocomplemented lattice of factor W*-subalgebras

[von Neumann'29, Murray–von Neumann'36]

- Given a W*-algebra $\mathcal{N}$, let $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{A}_2$ be W*-subalgebras of $\mathcal{N}$.
- A commutant of $\mathcal{A}$ in $\mathcal{N} := \mathcal{A}^{\blacksquare} := \{x \in \mathcal{N} : xy = yx \; \forall y \in \mathcal{A}\}$.
- $\mathcal{A}_1 \wedge \mathcal{A}_2 :=$ the largest W*-subalgebra of $\mathcal{N}$ contained in $\mathcal{A}_1$ and $\mathcal{A}_2$.
- $\mathcal{A}_1 \vee \mathcal{A}_2 :=$ the smallest W*-subalgebra of $\mathcal{N}$ containing $\mathcal{A}_1$ and $\mathcal{A}_2$.
- This implies:
  - $\mathcal{A}_1 \wedge \mathcal{A}_2 = \mathcal{A}_1 \cap \mathcal{A}_2$;
  - $(\mathcal{A}_1 \wedge \mathcal{A}_2)^{\blacksquare} = \mathcal{A}_1^{\blacksquare} \vee \mathcal{A}_2^{\blacksquare}$;
  - $\mathcal{N}^{\blacksquare} = \mathbb{C}\mathbb{I}$;
  - $\mathcal{A}_1 \vee \mathcal{A}_2 = (\mathcal{A}_1 \cup \mathcal{A}_2)^{\blacksquare\blacksquare}$;
  - $(\mathcal{A}_1 \vee \mathcal{A}_2)^{\blacksquare} = \mathcal{A}_1^{\blacksquare} \wedge \mathcal{A}_2^{\blacksquare}$;
  - $(\mathbb{C}\mathbb{I})^{\blacksquare} = \mathcal{N}$.
- $\mathcal{A}$ is a factor iff $\mathcal{A} \cap \mathcal{A}^{\blacksquare} = \mathbb{C}\mathbb{I}$ ($\iff \mathcal{A} \vee \mathcal{A}^{\blacksquare} = \mathcal{N}$).
- Hence: the set $\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N})$ of factor W*-subalgebras of a factor W*-algebra $\mathcal{N}$, equipped with $(\vee, \wedge, {}^{\blacksquare}, 0 := \mathbb{C}\mathbb{I}, 1 := \mathcal{N})$ is an orthocomplemented lattice.

Vacuum algebraic q.f.t.: Haag's «tentative postulate»

[Araki'61, Haag–Schroer'62, Haag–Kastler'64, Haag'92/'96]

- Typical* setting of algebraic q.f.t.: *(More general variants exhibit the same main ideas.)
  1) a functor $\mathfrak{A}$ from the category of subsets of space-time with embeddings as morphisms to the category of W*-subalgebras of a W*-algebra with embeddings as morphisms,
  2) if $S_1 \subseteq S_2$, then $\mathfrak{A}(S_1) \subseteq \mathfrak{A}(S_2)$,
  3) if $S = \bigcup_j S_j$, then $\mathfrak{A}(S) = \bigvee_j \mathfrak{A}(S_j) = (\bigcup_j \mathfrak{A}(S_j))^{\blacksquare\blacksquare}$ (=: additivity),
  4) if $S_1 \subseteq S_2^{\perp}$, then $\mathfrak{A}(S_1) \subseteq (\mathfrak{A}(S_2))^{\blacksquare}$ (=: causality).
- Haag–Schroer duality property := $(\mathfrak{A}(S^{\perp}) = (\mathfrak{A}(S))^{\blacksquare})$.
- [Haag'92/'96]**: «tentative postulate»: **(The '92 version differs from the '96 version.)
  1) consider an orthocomplemented lattice $(L_{(M,g)}, {}^{\perp})$, where (M, g) is a Minkowski space-time, and ${}^{\perp}$ is a time-like nonsignalling;
  2) consider an orthocomplemented lattice $(\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}), {}^{\blacksquare})$ of factor W*-subalgebras of a factor W*-algebra $\mathcal{N}$;
  3) an algebraic q.f.t., for the vacuum sector of the theory, is given by an orthocomplemented lattice homomorphism $(L_{(M,g)}, {}^{\perp}) \to (\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}), {}^{\blacksquare})$.
- In general, Haag's «tentative postulate» is too strong:
  1) the Haag–Schroer duality does not hold in several models;
  2) $\wedge$-preservation is usually not required and not verified.

Vacuum pre-a.q.f.t.: beyond Haag's «tentative postulate» [Kostecki'24]

- Let (M,g) be any lorentzian space-time, and $(L_{(M,g)},{}^{\perp})$ be a causal logic.
- Consider the following categorical reformulation, and weakening, of Haag's postulate:

  a vacuum pre-a.q.f.t. := a bounded join semilattice homomorphism that preserves all joins, $\mathfrak{N}^{\flat}:(L_{(M,g)},{}^{\perp})\to(\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}),{}^{\blacksquare})$, satisfying $\mathfrak{N}^{\flat}((\cdot)^{\perp})\leq(\mathfrak{N}^{\flat}(\cdot))^{\blacksquare}$ ("causality").

- We will say that $\mathfrak{N}^{\flat}$ satisfies the Haag–Schroer duality iff $\mathfrak{N}^{\flat}((\cdot)^{\perp})=(\mathfrak{N}^{\flat}(\cdot))^{\blacksquare}$.
- Since $(L_{(M,g)},{}^{\perp})$ is complete, by the adjoint functor theorem, there exists a bounded meet semilattice homomorphism that preserves all meets, $\mathfrak{N}^{\sharp}:(\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}),{}^{\blacksquare})\to(L_{(M,g)},{}^{\perp})$, that is right adjoint to $\mathfrak{N}^{\flat}$, i.e. $\mathfrak{N}^{\flat}\dashv\mathfrak{N}^{\sharp}$ is a monotone Galois connection:

$$\mathfrak{N}^{\flat}(x)\leq y \iff x\leq\mathfrak{N}^{\sharp}(y). \qquad (\spadesuit)$$

- All of this works also under generalisation to $((\mathrm{Sub}(M))_{\perp\perp},\subseteq,\cap,\curlyvee,\varnothing,M,{}^{\perp})$ (up to an additional assumption that it has all joins, required for $(\spadesuit)$ to hold).

Emergence in pre-a.q.f.t. (I) [Kostecki'25]

- Haag's «tentative postulate» of an orthocomplemented lattice homomorphism $H : (L_{(M,g)}, {}^{\perp}) \to (\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N})^{\blacksquare})$ contains implicitly a prescription of emergence of the causal logic structure from the structure of a factor W*-subalgebras for a sublattice of $(L_{(M,g)}, {}^{\perp})$ on which H is an isomorphism.
- The vacuum pre-a.q.f.t. $\mathfrak{N}^{\flat} : (L_{(M,g)}, {}^{\perp}) \to (\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}), {}^{\blacksquare})$, when combined with its right adjoint functor $\mathfrak{N}^{\sharp} : (\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}), {}^{\blacksquare}) \to (L_{(M,g)}, {}^{\perp})$, induces an equivalence of sublattices defined by $\{x \in L_{(M,g)} : \mathfrak{N}^{\sharp} \circ \mathfrak{N}^{\flat}(x) \leq x\}$ and $\{y \in \mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}) : y \leq \mathfrak{N}^{\flat} \circ \mathfrak{N}^{\sharp}(y)\}$, which can be seen as emergence of a part of a causal structure of a spacetime from the structure of the pre-a.q.f.t.:

$$\begin{tikzcd}
(L_{(M,g)}, {}^{\perp}) \arrow[rr, bend left, "\mathfrak{N}^{\flat}"] \arrow[rr, phantom, "\perp"] & & (\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}), {}^{\blacksquare}) \arrow[ll, bend left, "\mathfrak{N}^{\sharp}"] \\
\mathfrak{N}^{\sharp} \circ \mathfrak{N}^{\flat}(L_{(M,g)}, {}^{\perp}) \arrow[u, hook] \arrow[rr, "\cong"] & & \mathfrak{N}^{\flat} \circ \mathfrak{N}^{\sharp}(\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}), {}^{\blacksquare}). \arrow[u, hook]
\end{tikzcd}$$

- We will call it a strong emergence, if $\mathfrak{N}^{\flat}$ satisfies the Haag–Schroer duality.
- This holds also under generalisation from $L_{(M,g)}$ to $(\mathrm{Sub}(M))_{\perp\perp}$, provided the latter has all joins.

More general approach to a.q.f.t. (I)

- In general, a.q.f.t. models:
  - deal with valuations into W*-algebras $\mathcal{N}$ that are not necessarily factors;
  - allow domains given by sets X (equipped with a causal complement $^\perp$) that are not necessarily subsets of a space-time (M, g);
  - may exhibit limited availability of joins at their domain of definition.
- More precisely [Baumgärtel–Wollenberg'92]:
  - A set $\mathrm{Sub}(X) \subsetneq 2^X$ is called a causal index iff $(\mathrm{Sub}(X), \subseteq, ^\perp, \varnothing)$ is a quasiintuitionistic poset with a bottom element $\varnothing$, $Q^\perp \neq \varnothing$ $\forall Q \in \mathrm{Sub}(X)$, $\exists \{Q_i \in \mathrm{Sub}(X) \,:\, i \in \mathbb{N}\}$ such that $\bigcup_{i \in \mathbb{N}} Q_i = X$ and if $i \neq j$, then $Q_i \neq Q_j$ $\forall i, j \in \mathbb{N}$.
  - Let $\mathrm{Sub}_{\mathrm{W}^*}(\mathcal{N}) :=$ the set of W*-subalgebras of $\mathcal{N}$. An order monotone map (= a copresheaf) $\mathfrak{A} : (\mathrm{Sub}(X), \subseteq) \to (\mathrm{Sub}_{\mathrm{W}^*}(\mathcal{N}), \subseteq)$ is said to be:
    - causal iff if $Q_1 \subseteq {Q_2}^\perp$, then $\mathfrak{A}(Q_1) \subseteq (\mathfrak{A}(Q_2))^\blacksquare$;
    - additive (resp., finitely additive; countably additive) iff $\mathfrak{A}(\bigvee_{i \in I} Q_i) = \bigvee_{i \in I} \mathfrak{A}(Q_i)$ for all (resp., finite; countable) sets I and all $\{Q_i \in \mathrm{Sub}(X) \,:\, i \in I\}$ that are bounded from above and satisfy $\forall P \in \mathrm{Sub}(X)\ \exists R \in \mathrm{Sub}(X)\ \exists i \in I \,:$ if $P \subseteq \sup_{j \in I} Q_j$ and $R \subseteq Q_i$, then $R \subseteq P$.
  - For examples of additive causal copresheaves on suitable causal index subsets of globally hyperbolic (M, g) see, e.g., [Fewster–Verch'12'15].

More general approach to a.q.f.t. (II)

- In order to obtain the join semilattice structure from the causal index set, it has to be constructed by means of a join-extension.
- [Keyl'98]: If $(\mathrm{Sub}(X), \subseteq, {}^{\perp}, \varnothing)$ is join-extended to a bounded join semilattice $(\widetilde{\mathrm{Sub}}(X), \subseteq, \vee, {}^{\perp}, \varnothing, X)$, defined by
  $\widetilde{\mathrm{Sub}}(X) := \mathrm{Sub}(X) \cup \{\bigvee_{i\in I} Q_i \,:\, Q_i \in \mathrm{Sub}(X), i \in I$, and I is arbitrary (resp., finite; countable) set$\}$, then an additive causal copresheaf $\mathfrak{A}$ on $\mathrm{Sub}(X)$ extends to a bounded join semilattice homomorphism $\widetilde{\mathfrak{A}} : \widetilde{\mathrm{Sub}}(X) \to \mathrm{Sub}_{\mathrm{W}^*}(\mathcal{N})$ which preserves all (resp., finite; countable) joins.
- The inclusion $\mathrm{Sub}(X) \subseteq \widetilde{\mathrm{Sub}}(X)$ preserves all existing meets. However $\widetilde{\mathrm{Sub}}(X)$ may contain additional meets. So, if one wants $\widetilde{\mathrm{Sub}}(X)$ to be quasiintuitionistic (i.e. $Q \cap Q^{\perp} = 0 \;\forall Q \in \widetilde{\mathrm{Sub}}(X)$), then this has to be assumed additionally.

A quasiminimal perspective on pre-a.q.f.t. (I) [Kostecki'26]

- Consider a set X and a bounded quasiminimal join semilattice $(\widetilde{\mathrm{Sub}}(X), \subseteq, \vee, \varnothing, X, {}^{\perp})$ that is a finite join-extension of an additive causal index poset $(\mathrm{Sub}(X), \subseteq, \varnothing, {}^{\perp})$.
- an action of the closure operator ${}^{\perp\perp}$ on this semilattice results in a bounded involutive join semilattice $((\widetilde{\mathrm{Sub}}(X))_{\perp\perp}, \subseteq, \curlyvee, \varnothing, X, {}^{\perp})$.
- If the condition $x \wedge x^{\blacksquare} = 0\ \forall x \in \mathcal{A}$ is dropped off, then the unital *-subalgebras $\mathcal{A}$ of a given W*-algebra $\mathcal{N}$ are no longer factors, and $(\mathrm{Sub}_{\mathrm{u}^*}(\mathcal{N}), \subseteq, \wedge, \vee, \mathbb{C}\mathbb{I}, \mathcal{N}, {}^{\blacksquare})$ is a bounded quasiminimal algebra, with an internal bounded involutive lattice of W*-subalgebras $(\mathrm{Sub}_{\mathrm{W}^*}(\mathcal{N}), \subseteq, \wedge, \cdot \curlyvee \cdot := (\cdot \vee \cdot)^{\blacksquare\blacksquare}, \mathbb{C}\mathbb{I}, \mathcal{N}, {}^{\blacksquare})$.
- Taking causal copresheaf $\mathfrak{A}$ over $\mathrm{Sub}(X)$ to be valued in the unital *-subalgebras of a W*-algebra $\mathcal{N}$, and denoting its finitely additive join-extension by $\widetilde{\mathfrak{A}}$, we arrive at the commutative diagram of bounded join semilattice homomorphisms:

$$\begin{tikzcd}
(\widetilde{\mathrm{Sub}}(X), \subseteq, \vee, \varnothing, X, {}^{\perp}) \arrow[r, "\widetilde{\mathfrak{A}}"] \arrow[d, "(\cdot)^{\perp\perp}"'] & (\mathrm{Sub}_{\mathrm{u}^*}(\mathcal{N}), \subseteq, \wedge, \vee, {}^{\blacksquare}, \mathbb{C}\mathbb{I}, \mathcal{N}) \arrow[d, "(\cdot)^{\blacksquare\blacksquare}"] \\
((\widetilde{\mathrm{Sub}}(X))_{\perp\perp}, \subseteq, \curlyvee, \varnothing, X, {}^{\perp}) \arrow[r, "\widetilde{\mathfrak{A}}"] & (\mathrm{Sub}_{\mathrm{W}^*}(\mathcal{N}), \subseteq, \wedge, \curlyvee, {}^{\blacksquare}, \mathbb{C}\mathbb{I}, \mathcal{N}),
\end{tikzcd}$$

A quasiminimal perspective on pre-a.q.f.t. (II) [Kostecki'26]

- If $(\widetilde{\mathrm{Sub}}(X), \subseteq, \vee, \varnothing, X, {}^{\perp})$ is assumed to be also quasiintuitionistic (i.e. $Q \cap Q^{\perp} = 0 \; \forall Q \in \widetilde{\mathrm{Sub}}(X)$), then $(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}$ also satisfies this condition (however, it does not have to be an orthocomplemented lattice, since arbitrary two elements may not have a meet).
- If also the same is assumed on $\mathrm{Sub}_{\mathrm{u}^*}(\mathcal{N})$, restricting it to the set $\mathrm{Sub}_{\mathrm{fu}^*}(\mathcal{N})$ of unital factor *-subalgebras of $\mathcal{N}$, then the above commutative diagram restricts to

$$\begin{tikzcd}
(\widetilde{\mathrm{Sub}}(X), \subseteq, \vee, \varnothing, X, {}^{\perp}) \arrow[r, "\widetilde{\mathfrak{A}}"] \arrow[d, "(\cdot)^{\perp\perp}"'] & (\mathrm{Sub}_{\mathrm{fu}^*}(\mathcal{N}), \subseteq, \wedge, \vee, {}^{\blacksquare}, \mathbb{C}\mathbb{I}, \mathcal{N}) \arrow[d, "(\cdot)^{\blacksquare\blacksquare}"] \\
((\widetilde{\mathrm{Sub}}(X))_{\perp\perp}, \subseteq, \curlyvee, \varnothing, X, {}^{\perp}) \arrow[r, "\widetilde{\mathfrak{A}}"] & (\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}), \subseteq, \wedge, \curlyvee, {}^{\blacksquare}, \mathbb{C}\mathbb{I}, \mathcal{N}).
\end{tikzcd}$$

- On the other hand, if $(\widetilde{\mathrm{Sub}}(X),\subseteq,\vee,\varnothing,X,{}^{\perp})$ is a join-extension of $(\mathrm{Sub}(X),\subseteq,\vee,\varnothing,{}^{\perp})$ with all joins, and $\widetilde{\mathfrak{A}}^{\flat}$ is a corresponding additive join-extension of an additive causal copresheaf $\mathfrak{A}$, then, by the adjoint functor theorem for posets, there exist a right adjoint map $\widetilde{\mathfrak{A}}^{\sharp}$ that preserves all meets.
- Let $\mathrm{j}_{\perp\perp}:(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}\to\widetilde{\mathrm{Sub}}(X)$ and $\mathrm{j}_{\blacksquare\blacksquare}:\mathrm{Sub}_{\mathrm{W}^*}(\mathcal{N})\to\mathrm{Sub}_{\mathrm{u}^*}(\mathcal{N})$ denote embeddings.
- In such case the adjunctions

$$\begin{tikzcd}
(\widetilde{\mathrm{Sub}}(X),\subseteq,\vee,\varnothing,X,{}^{\perp}) \arrow[r, bend left, "\widetilde{\mathfrak{A}}^{\flat}"] \arrow[r, phantom, "\perp"] \arrow[d, bend left, "(\cdot)^{\perp\perp}"] \arrow[d, phantom, "\dashv"] & (\mathrm{Sub}_{\mathrm{u}^*}(\mathcal{N}),\subseteq,\wedge,\vee,{}^{\blacksquare},\mathbb{C}\mathbb{I},\mathcal{N}) \arrow[l, bend left, "\widetilde{\mathfrak{A}}^{\sharp}"] \arrow[d, bend left, "(\cdot)^{\blacksquare\blacksquare}"] \arrow[d, phantom, "\dashv"] \\
((\widetilde{\mathrm{Sub}}(X))_{\perp\perp},\subseteq,\curlyvee,\varnothing,X,{}^{\perp}) \arrow[u, bend left, "\mathrm{j}_{\perp\perp}"] \arrow[r, bend left, "\widetilde{\mathfrak{A}}^{\flat}"] \arrow[r, phantom, "\perp"] & (\mathrm{Sub}_{\mathrm{W}^*}(\mathcal{N}),\subseteq,\wedge,\curlyvee,{}^{\blacksquare},\mathbb{C}\mathbb{I},\mathcal{N}) \arrow[l, bend left, "\widetilde{\mathfrak{A}}^{\sharp}"] \arrow[u, bend left, "\mathrm{j}_{\blacksquare\blacksquare}"]
\end{tikzcd}$$

induce the "emergent" equivalences:

$$\widetilde{\mathfrak{A}}^{\sharp}\circ\widetilde{\mathfrak{A}}^{\flat}((\widetilde{\mathrm{Sub}}(X))_{\perp\perp})\cong\widetilde{\mathfrak{A}}^{\flat}\circ\widetilde{\mathfrak{A}}^{\sharp}((\mathrm{Sub}_{\mathrm{W}^*}(\mathcal{N}))),$$

$$\{x\in(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}:(\cdot)^{\perp\perp}\circ\widetilde{\mathfrak{A}}^{\sharp}\circ\widetilde{\mathfrak{A}}^{\flat}\circ\mathrm{j}(x)\leq x\}\cong\{y\in\mathrm{Sub}_{\mathrm{u}^*}(\mathcal{N}):y\leq\widetilde{\mathfrak{A}}^{\flat}\circ\mathrm{j}\circ(\cdot)^{\perp\perp}\circ\widetilde{\mathfrak{A}}^{\sharp}(y)\},$$

which are "strong" iff $\widetilde{\mathfrak{A}}^{\flat}$ satisfies the Haag–Schroer duality on $(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}$ (resp., $\widetilde{\mathrm{Sub}}(X)$).
- Analogous result holds under restriction to quasiintuitionistic case.

Spectral presheaf vacuum pre-a.q.f.t [Kostecki'24'26]

- When $((\widetilde{\mathrm{Sub}}(X))_{\perp\perp}, \subseteq, \cap, \curlyvee, \varnothing, X, {}^{\perp})$ is an orthocomplemented lattice (e.g., if $(\widetilde{\mathrm{Sub}}(X))_{\perp\perp} = L_{(M,g)}$), then it admits a join preserving embedding $\delta^{\wedge}$ into the spectral presheaf $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^{(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}})$. If it also has all joins, then $\bullet_{\perp} := \delta^{\wedge} \circ {}^{\perp} \circ \varepsilon^{\wedge}$ is well defined.
- We define a spectral presheaf vacuum pre-a.q.f.t. as a bounded join semilattice homomorphism of (0,1)-Akchurin algebras

$$\mathfrak{M}^{\flat} : (\mathrm{Sub}_{\mathrm{clop}}(\Sigma^{(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}}), \bullet_{\perp}) \to (\mathrm{Sub}_{\mathrm{clop}}(\Sigma^{\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N})}), \bullet_{\blacksquare}),$$

such that:
  1) $\mathfrak{M}^{\flat}((\cdot)^{\bullet_{\perp}}) \leq (\mathfrak{M}^{\flat}(\cdot))^{\bullet_{\blacksquare}}$ ("spectral presheaf causality");
  2) the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Sub}_{\mathrm{clop}}(\Sigma^{(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}}) \arrow[r, "\mathfrak{M}^{\flat}"] & \mathrm{Sub}_{\mathrm{clop}}(\Sigma^{\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N})}) \\
(\widetilde{\mathrm{Sub}}(X))_{\perp\perp} \arrow[u, "\delta^{\wedge}"] \arrow[r, "\widetilde{\mathfrak{A}}^{\flat}"] & \mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}). \arrow[u, "\delta^{\wedge}"']
\end{tikzcd}$$

- We will say that a spectral presheaf vacuum pre-a.q.f.t. $\mathfrak{M}^{\flat}$ satisfies the spectral presheaf Haag–Schroer duality iff $\mathfrak{M}^{\flat}((\cdot)^{\bullet_{\perp}}) = (\mathfrak{M}^{\flat}(\cdot))^{\bullet_{\blacksquare}}$.

Paraconsistent borders + Emergence in spectral presheaf pre-a.q.f.t.

[Kostecki'19'24'25'26]

- If $S \in (\widetilde{\mathrm{Sub}}(X))_{\perp\perp} \setminus \{\varnothing, X\}$ and $\mathcal{A} \in \mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}) \setminus \{\mathbb{C}\mathbb{I}, \mathcal{N}\}$, then $\partial_{\bullet_\perp} \delta^\wedge(S) \neq 0$ and $\partial_{\bullet_\blacksquare} \delta^\wedge(\mathcal{A}) \neq 0$ are, respectively, the borders of a causally closed set S and of a factor W*-subalgebra $\mathcal{A}$ (this addresses, in a particular way, the issue raised by [Nobili'06]).
- $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^{(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}})$ is complete, so $\mathfrak{M}^\flat$ has the (unique) right adjoint functor $\mathfrak{M}^\sharp : \mathrm{Sub}_{\mathrm{clop}}(\Sigma^{\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N})}) \to \mathrm{Sub}_{\mathrm{clop}}(\Sigma^{(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}})$.
- The adjunctions

$$\begin{tikzcd}
\mathrm{Sub}_{\mathrm{clop}}(\Sigma^{(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}}) \arrow[rr, bend left, "\mathfrak{M}^\flat"] \arrow[rr, phantom, "\perp"] \arrow[d, bend left, "\varepsilon^\wedge"] \arrow[d, phantom, "\dashv"] & & \mathrm{Sub}_{\mathrm{clop}}(\Sigma^{\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N})}) \arrow[ll, bend left, "\mathfrak{M}^\sharp"] \arrow[d, bend left, "\varepsilon^\wedge"] \arrow[d, phantom, "\dashv"] \\
(\widetilde{\mathrm{Sub}}(X))_{\perp\perp} \arrow[u, bend left, "\delta^\wedge"] \arrow[rr, bend left, "\widetilde{\mathfrak{A}}^\flat"] \arrow[rr, phantom, "\perp"] & & \mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}) \arrow[u, bend left, "\delta^\wedge"] \arrow[ll, bend left, "\widetilde{\mathfrak{A}}^\sharp"]
\end{tikzcd}$$

  induce the "emergent" equivalence:

$$\{x \in (\widetilde{\mathrm{Sub}}(X))_{\perp\perp} : \varepsilon^\wedge \circ \mathfrak{M}^\sharp \circ \mathfrak{M}^\flat \circ \delta^\wedge(x) \leq x\} \cong$$
$$\cong \{y \in \mathrm{Sub}_{\mathrm{clop}}(\Sigma^{\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N})}) : y \leq \mathfrak{M}^\flat \circ \delta^\wedge \circ \varepsilon^\wedge \circ \mathfrak{M}^\sharp(y)\},$$

  which is "strong" iff $\mathfrak{M}^\flat$ satisfies the spectral presheaf Haag–Schroer duality.

Cospectral lattice vacuum pre-a.q.f.t [Kostecki'26]

- On the other hand, given $\widetilde{\mathfrak{A}}^\flat : (\widetilde{\mathrm{Sub}}(X))_{\perp\perp} \to \mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N})$, if $((\widetilde{\mathrm{Sub}}(X))_{\perp\perp}, \subseteq, \cap, \curlyvee, \varnothing, X, {}^\perp)$ is an orthocomplemented lattice that has all meets, then the cospectral lattice vacuum pre-a.q.f.t. is defined as a bounded join semilattice homomorphism $\mathfrak{R}^\flat : \mathfrak{S}^{\mathrm{op}}((\widetilde{\mathrm{Sub}}(X))_{\perp\perp}) \to \mathfrak{S}^{\mathrm{op}}(\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}))$ such that $\mathfrak{R}^\flat((\cdot)^{\square_\perp}) \leq (\mathfrak{R}^\flat(\cdot))^{\square\blacksquare}$ and the following diagram of bounded join semilattice homomorphisms commutes:

$$\begin{tikzcd}
\mathfrak{S}^{\mathrm{op}}(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}) \arrow[r, "\mathfrak{R}^\flat"] \arrow[d, "(\cdot)^{\square_\perp\square_\perp}"] & \mathfrak{S}^{\mathrm{op}}(\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N})) \arrow[d, "(\cdot)^{\square\blacksquare\square\blacksquare}"'] \\
(\widetilde{\mathrm{Sub}}(X))_{\perp\perp} \arrow[r, "\widetilde{\mathfrak{A}}^\flat"] & \mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}).
\end{tikzcd}$$

- $((\widetilde{\mathrm{Sub}}(X))_{\perp\perp} = (\mathfrak{S}^{\mathrm{op}}(\widetilde{\mathrm{Sub}}(X))_{\perp\perp}))_{\square_\perp\square_\perp}, (\cdot)^\perp = (\cdot)^{\square_\perp})$ and $(\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N}) = (\mathfrak{S}^{\mathrm{op}}(\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N})))_{\square\blacksquare\square\blacksquare}, (\cdot)^{\blacksquare} = (\cdot)^{\square\blacksquare})$.
- All of the arrows in the above diagram admit adjoints, resulting with the corresponding "emergent" equivalence.
- An interesting open question: which join-extensions of causal index sets and which W*-algebras satisfy, respectively, $\mathfrak{S}^{\mathrm{op}}((\widetilde{\mathrm{Sub}}(X))_{\perp\perp}) = \widetilde{\mathrm{Sub}}(X)$ and $\mathfrak{S}^{\mathrm{op}}(\mathrm{Sub}_{\mathrm{fW}^*}(\mathcal{N})) = \mathrm{Sub}_{\mathrm{fu}^*}(\mathcal{N})$?

References

1) Kostecki R.-P., 2024, *Paraconsistency of relativistic nonsignalling, and some other features of causal spectral toposes*, talk at: Causalworlds 2024, Perimeter Institute, Waterloo, https://pirsa.org/24090141.
2) Kostecki R.-P., 2025, *Topos-theoretic extension of vacuum algebraic quantum field theory over curved space-times*, talk at: Jerzy Lewandowski memorial conference, Banach Center, Warszawa, https://jurekmemorial.fuw.edu.pl/slides/kostecki.pdf.
3) Engel B., Kostecki R.-P., 2026, *Four negations and the spectral presheaf*, https://www.arxiv.org/pdf/2603.08699. (A major revision is coming soon.)
4) Kostecki R.-P., 2026, *Causal logics revisited*, at: Observers and causality in quantum gravity, Research Center for Quantum Information, Bratislava, https://www.fuw.edu.pl/∼kostecki/poster_2026.04.21_bratislava.pdf.

In preparation:

5) Engel B., Kostecki R.-P., 2026, *Biquasiintuitionistic algebras and spectral presheaves*.
6) Kostecki R.-P., 2026, *Causal logics re-revisited*.
7) Kostecki R.-P., 2026, *Algebraic perspective on the Vakarelov–Sotirov–Došen extension of the full nonassociative Orlov–Lambek–Grishin calculus*.