Introduction to the model theory of modules and its applications in the representation theory

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Plan of the talk

1. Fundamentals of the model theory of modules.
2. Basics on the representation theory of finite dimensional algebras. Prest's conjecture on the Krull-Gabriel dimension.
3. The main results: applications of covering theory of quivers and functor categories to the Prest's conjecture.

1.1 Pp-formulas

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of R on M ($m \cdot r \in M$, for any $r \in R$ and $m \in M$).

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  of right R-modules.
- The set of all formulas over Σ_R is denoted by $Form(R)$.

1.1 Pp-formulas

Definition

Assume that $A \in \mathbb{M}_{l,t}(R)$ and $B \in \mathbb{M}_{k,t}(R)$. A **pp-formula** (**positive primitive formula**) is a formula in *Form*(R) of the form:

$$\varphi(v_1, \dots, v_k) = \exists_{x_1,\dots,x_l} \, [x_1, \dots, x_l] \cdot A = [v_1, \dots, v_k] \cdot B.$$

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Theorem (Baur-Monk, 1969)

Every formula in $Form(R)$ is equivalent with some boolean combination of pp-formulas. $\square$

1.2 Algebraically compact modules

Definition

- For any pp-formula $\varphi(v_1, \ldots, v_k)$, the set

$$\varphi(M) := \{\underline{m} \in M^k \mid M \models \varphi(\underline{m})\}$$

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- A module M is **algebraically compact** if any descending chain of nonzero pp-definable subgroups of M^k, for any k, is nonempty. Equivalently, any system of R-linear equations

$$\sum_{j=1}^{n_i} x_{ij} \cdot r_{ij} = m_i,$$

  $i \in I$ (I **possibly infinite**), $r_{ij} \in R$, $m_i \in M$ which is finitely solvable in M, is actually solvable in M.

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One can show that any R-module is elementary equivalent with an algebraically compact module.

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A monomorphism $f : M \hookrightarrow N$ of R-modules is **pure** if

$$M \models \varphi(m_1, \ldots, m_k) \Leftrightarrow N \models \varphi(f(m_1), \ldots, f(m_k)),$$

for any $k \in \mathbb{N}$, pp-formula $\varphi(v_1, \ldots, v_k)$ and $m_1, \ldots, m_k \in M$, i.e. <u>every system of R-linear equations, with coefficients in M, which is solvable in N is already solvable in a submodule M of N</u>.

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Hence:

- M **is pure-injective** if for any pure mono $f : M \hookrightarrow N$ the induced exact sequence $0 \to M \xrightarrow{f} N \to L \to 0$ splits. Equivalently, $f : M \hookrightarrow N$ is a section $M \hookrightarrow M \oplus M' \cong N$.

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Warfield showed that any R-module M has the **pure-injective envelope** $\mathcal{E}(M)$ which is the smallest pure-injective module such that $M \subseteq \mathcal{E}(M)$ and the embedding $M \hookrightarrow \mathcal{E}(M)$ is pure.

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Theorem (Gabriel-Oberst, 1966)

Assume that M is a pure-injective R-module and let $\{M_\lambda\}_{\lambda \in \Lambda}$ be the set of all indecomposable direct summands of M. Then

$$M \cong \mathcal{E}(\bigoplus_{\lambda \in \Lambda} M_\lambda) \oplus M'$$

where M' is pure-injective **without indecomposable direct summands**, i.e. **super-decomposable**. This decomposition is unique up to isomorphism and permutation of direct summands. $\square$

Rings for which there are no super-decomposable pure-injectives have the decomposition theory of pure-injectives much simpler.

1.3 Krull-Gabriel dimension

- $\mathrm{mod}(R)$ is the category of all finitely presented R-modules, i.e. the modules M for which there is an exact sequence

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- $\mathrm{MOD}(\mathrm{mod}(R))$ is the category of all additive K-linear contravariant functors $\mathrm{mod}(R) \to \mathrm{MOD}(K)$.
- $\mathcal{F}(R)$ is the full subcategory of $\mathrm{MOD}(\mathrm{mod}(R))$ formed by **finitely presented functors**, i.e. $T \in \mathcal{F}(R)$ if there is an exact sequence of functors

$${}_R(-, M) \xrightarrow{{}_R(-,f)} {}_R(-, N) \to T \to 0,$$

  for $M, N, f : M \to N \in \mathrm{mod}(R)$, so $T \cong \mathrm{Coker}_R(-, f)$.

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  for $M, N, f : M \to N \in \mathrm{mod}(R)$, so $T \cong \mathrm{Coker}_R(-, f)$.
- Categories $\mathrm{MOD}(\mathrm{mod}(R))$ and $\mathcal{F}(R)$ are abelian.

1.3 Krull-Gabriel dimension

Assume that $\mathcal{C}$ is a skeletally small abelian category.

- A full subcategory $\mathcal{S}$ of $\mathcal{C}$ is a **Serre subcategory**, if $\mathcal{S}$ is closed under subobjects, quotients and extensions.

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- The **Krull-Gabriel filtration** $(\mathcal{C}_\alpha)_\alpha$ of $\mathcal{C}$ is <u>some special filtration of $\mathcal{C}$</u> by Serre subcategories $\mathcal{C}_\alpha$ which is defined recursively and indexed by all ordinal numbers.

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- The **Krull-Gabriel dimension** $\mathrm{KG}(\mathcal{C})$ of $\mathcal{C}$ is the smallest ordinal number α such that $\mathcal{C}_\alpha = \mathcal{C}$, if it exists. Otherwise, set $\mathrm{KG}(\mathcal{C}) = \infty$ and say that KG-dim is **undefined**.

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- The **Krull-Gabriel dimension** $\mathrm{KG}(\mathcal{C})$ of $\mathcal{C}$ is the smallest ordinal number α such that $\mathcal{C}_\alpha = \mathcal{C}$, if it exists. Otherwise, set $\mathrm{KG}(\mathcal{C}) = \infty$ and say that KG-dim is **undefined**.
- **We set** $\mathrm{KG}(R) := \mathrm{KG}(\mathcal{F}(R))$.

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Remark

If $\mathrm{KG}(R)$ exists, then there is no super-decomposable pure-injective R-module. (The converse is not known in general.)
Hence in this case, the decomposition theory in the class of pure-injective modules is much simpler.

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Aim

We are interested in determining whether $\mathrm{KG}(R)$ exists and what are its possible values in the very special situation when R is a **finite dimensional** K**-algebra** A over an algebraically closed field K (hence A is a ring with a compatible K-vector space structure).
This leads to the very rich field of

Representation theory of finite dimensional algebras

2.1 Basics of representation theory

Main contributors to the field:

- **Foundations** (60'-80'): Pierre Gabriel (University of Zurich, Switzerland), Maurice Auslander (Brandeis University, USA), Idun Reiten (NTNU, Norway), Andrei V. Roiter, Yuriy Drozd (National Academy of Sciences, Ukraine).
- **Golden Age** (70'-2000'): Claus Michael Ringel, Dieter Happel, Henning Krause (Bielefeld University, Germany), Helmut Lenzing (University of Paderborn, Germany), Bernhard Keller (Universite Paris Cite, France) Daniel Simson, Andrzej Skowroński (Nicolaus Copernicus University, Poland), William Crawley-Boevey (University of Leeds, UK), Andrei Zelevinsky (Northeastern University).
- **Nowadays**: some from above, Osamu Iyama (University of Tokio, Japan), Lidia Angeleri-Huegel (University if Verona, Italy), Jan Šťovíček (Charles University, Czech Republic), many more...

2.1 Basics of representation theory

Assume that A is a finite dimensional K-algebra over $\overline{K} = K$.

Aim of the representation theory

Classify all modules in $\mathrm{mod}(A)$ and all homomorphisms between them. By the Krull-Remak-Schmidt theorem, this task boils down to the classification of all **indecomposable** A**-modules** ($\mathrm{ind}(A)$ denotes the category of indecomposable A-modules).

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Main tools:

- **quiver techniques** (Gabriel, 70'),
- **almost split sequences** (Auslander-Reiten, 70'-80'),
- other notable: tilting theory, covering theory of quivers, (homological algebra, triangulated categories, algebraic geometry, commutative algebra, combinatorics), many more.

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Quiver techniques combined with theory of almost split sequences allow to **visualize the category $\mathrm{ind}(A)$ in a convenient way**.

2.2 Quivers, path algebras and representations

Theorem (Gabriel, 1973)

Any (basic and indecomposable) finite dimensional K-algebra over $K = \overline{K}$ is isomorphic with a **bound quiver** K-**algebra**. $\square$

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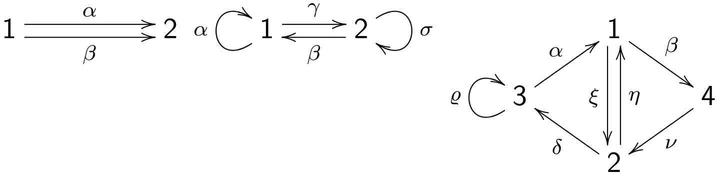

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- **Admissible ideals** in KQ are important. If I is admissible, then KQ/I is a **bound quiver algebra** associated with the **bound quiver** (Q, I).

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- The bound quiver (Q, I) also gives rise to the **bound quiver** K-**category** $\underline{KQ}/I$ whose objects are the vertices of Q and the morphism space from x to y is defined as $\mathcal{P}_Q(y, x)/I$.

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  - **representations** $(V_x, V_\alpha)_{x \in Q_0, \alpha \in Q_1}$ (Q_0 is the set of vertices of Q and Q_1 the set of arrows) where $V_x \in \mathrm{mod}(K)$ and $V_\alpha : V_x \to V_y$ for $\alpha : x \to y$; such a representation must satisfy relations from I, in a sense.

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Remark

Gabriel invented the language of quivers by studying the filtration on the category $\mathcal{F}(R)$, now known as the **Krull-Gabriel filtration**.

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Assume that A is an algebra.

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Theorem (Gabriel, 1972; Donovan–Freislich, 1973)

A path algebra $A = KQ$ is of finite type if and only if Q is a **Dynkin quiver**. If A is of infinite type, then A is tame if and only if Q is an **Euclidean quiver**. $\square$

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Theorem (Auslander, 1982)

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Aim

Study Prest's conjecture with the **covering theory of quivers** as the main tool. We will mainly use the **Galois coverings**.

3.1 Covering theory: an example of a Galois covering

$$\begin{tikzcd}
 & & a_{-1} \arrow[dl, "\beta_{-1}"'] \arrow[dr, "\alpha_{-1}"] & & a_0 \arrow[dl, "\beta_0"'] \arrow[dr, "\alpha_0"] & & a_1 \arrow[dl, "\beta_1"'] \arrow[dr, "\alpha_1"] & \\
\widetilde{Q} = & \dots b_{-1} & & b_0 & & b_1 & & b_2 \dots
\end{tikzcd}$$

$$F \Big\downarrow$$

$$Q = a \underset{\alpha}{\overset{\beta}{\rightrightarrows}} b$$

- There is a quiver morphism $F : \widetilde{Q} \to Q$ such that $F(a_i) = a$, $F(b_i) = b$, $F(\alpha_i) = \alpha$, $F(\beta_i) = \beta$, for any $i \in \mathbb{Z}$.

3.1 Covering theory: an example of a Galois covering

$$\widetilde{Q} = \begin{tikzcd} & a_{-1} \arrow[dl, "\beta_{-1}"'] \arrow[dr, "\alpha_{-1}"] & & a_0 \arrow[dl, "\beta_0"'] \arrow[dr, "\alpha_0"] & & a_1 \arrow[dl, "\beta_1"'] \arrow[dr, "\alpha_1"] & \\ \ldots b_{-1} & & b_0 & & b_1 & & b_2 \ldots \end{tikzcd}$$

$$\Big\downarrow F$$

$$Q = \begin{tikzcd} a \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\alpha"'] & b \end{tikzcd}$$

- There is a quiver morphism $F : \widetilde{Q} \to Q$ such that $F(a_i) = a$, $F(b_i) = b$, $F(\alpha_i) = \alpha$, $F(\beta_i) = \beta$, for any $i \in \mathbb{Z}$.
- For any $i \in \mathbb{Z}$ there is a bijection between arrows starting in a_i (ending in b_i) and starting in $F(a_i) = a$ (ending in $F(b_i) = b$).

3.1 Covering theory: an example of a Galois covering

$$
\begin{tikzcd}[column sep=small]
 & & a_{-1} \arrow[dl, "\beta_{-1}"'] \arrow[dr, "\alpha_{-1}"] & & a_0 \arrow[dl, "\beta_0"'] \arrow[dr, "\alpha_0"] & & a_1 \arrow[dl, "\beta_1"'] \arrow[dr, "\alpha_1"] & \\
\widetilde{Q} = & \dots b_{-1} & & b_0 & & b_1 & & b_2 \dots
\end{tikzcd}
$$

$$
\Big\downarrow F
$$

$$
\begin{tikzcd}
Q = a \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\alpha"'] & b
\end{tikzcd}
$$

- There is a quiver morphism $F : \widetilde{Q} \to Q$ such that $F(a_i) = a$, $F(b_i) = b$, $F(\alpha_i) = \alpha$, $F(\beta_i) = \beta$, for any $i \in \mathbb{Z}$.
- For any $i \in \mathbb{Z}$ there is a bijection between arrows starting in a_i (ending in b_i) and starting in $F(a_i) = a$ (ending in $F(b_i) = b$).
- The group $G = \mathbb{Z}$ acts on $\widetilde{Q}$ by shift and $\widetilde{Q}/G \cong Q$.

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- There is a quiver morphism $F : \widetilde{Q} \to Q$ such that $F(a_i) = a$, $F(b_i) = b$, $F(\alpha_i) = \alpha$, $F(\beta_i) = \beta$, for any $i \in \mathbb{Z}$.
- For any $i \in \mathbb{Z}$ there is a bijection between arrows starting in a_i (ending in b_i) and starting in $F(a_i) = a$ (ending in $F(b_i) = b$).
- The group $G = \mathbb{Z}$ acts on $\widetilde{Q}$ by shift and $\widetilde{Q}/G \cong Q$.
- $F : \widetilde{Q} \to Q$ induces a K-linear functor $F : K\widetilde{Q} \to KQ$.

3.2 Main results: covering theory of functor categories

Theorem (P., 2025)

The following assertions hold for a Galois G-covering $F : R \to A$.

(1) There exists a K-linear functor $\Phi : \mathcal{F}(R) \to \mathcal{F}(A)$ which is <u>exact and faithful</u>.

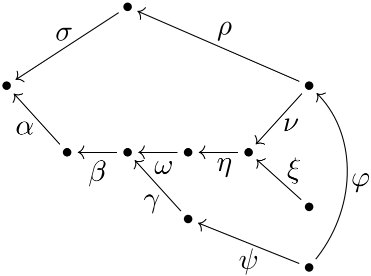
(2) We have $\mathrm{KG}(R) \leqslant \mathrm{KG}(A)$, i.e.

Galois coverings do not increase the Krull-Gabriel dimension.

(3) If the group G is torsion-free, then $\Phi : \mathcal{F}(R) \to \mathcal{F}(A)$ is a **Galois G-precovering of functor categories**.

3.3 Application to the Prest's conjecture

Let $A = KQ/I$ where

$Q =$ 

and $I = \langle \gamma\beta\alpha, \xi\eta\omega, \psi\gamma\beta - \varphi\nu\eta\omega\beta \rangle$.

- By [Bobiński, Dräxler, Skowroński, 2002] A is domestic with infinitely many components of type $\mathbb{Z}\mathbb{D}_\infty$ and $\mathbb{Z}\mathbb{A}_\infty^\infty$.
- Take a chosen Galois covering $F : R \to A$. If we have $\mathrm{KG}(R) = \infty$, then $\mathrm{KG}(A) = \infty$, because $\mathrm{KG}(R) \leqslant \mathrm{KG}(A)$. If this is the case, then Prest's conjecture fails.