A Double-Categorical Perspective (or two) on Dual Fibrations

Based on j.w.i.p with Rose Kudzman-Blais

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Motivation

- What kinds of double category are suitable to model various fragments of (and full) classical first-order logic (and also linear logic)?

- Progress:
  * Regular logic $\longrightarrow$ Nasu 2025
  * Coherent logic $\longrightarrow$ M. 2026 (upcoming)

- Today: progress towards adding negation/understanding duality

(De Morgan) Dualities in Logic

- Classical first-order logic
  * self-dual, explosive
- Full classical linear logic
  * self-dual, paraconsistent
- (Dual) intuitionistic logic
  * not self-dual, dual is paraconsistent

- in fibrations/hyperdoctrines, duality manifests as the dual fibration

- Goal: extend the $\mathbb{B}$il construction to lift this duality from fibrations to double categories (esp. equipments), to better understand duality/negation in double-categorical context

Double Categories

Definition A (pseudo) double category $\mathbb{D}$ is a pseudocategory internal to Cat:

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{c} \mathbb{D}_1 \underset{t}{\overset{s}{\underset{\xleftarrow{i}}{\rightrightarrows}}} \mathbb{D}_0$$

Explicitly, $\mathbb{D}$ consists of:

(i) objects $A, B, C, D, \ldots$ (objects of $\mathbb{D}_0$);

(ii) tight arrows $f : A \to B$ (morphisms of $\mathbb{D}_0$);

(iii) loose arrows $R : A \nrightarrow C$ (objects of $\mathbb{D}_1$);

(iv) cells
$$\begin{array}{ccc} A & \overset{R}{\nrightarrow} & C \\ {\scriptstyle f}\downarrow & \varphi & \downarrow{\scriptstyle g} \\ B & \underset{S}{\nrightarrow} & D \end{array}$$
(morphisms of $\mathbb{D}_1$).

Associativity and unitality for composition in the tight direction are on the nose; in the loose direction, they hold only up to coherent iso.

Double Categories (cont'd)

Examples

- $\mathbb{Q}(K)$ - quintets in a 2-category K
- $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ - sets, functions, spans
- $\mathbb{P}\mathrm{rof}$ - categories, functors, profunctors
- $\mathbb{R}\mathrm{el}^{\circ}$ and $\mathbb{R}\mathrm{el}^{\bullet}$ - sets, functions, relations, +

	$\mathbb{R}\mathrm{el}^{\circ}$	$\mathbb{R}\mathrm{el}^{\bullet}$
$\begin{array}{ccc} A & \overset{R}{\nrightarrow} & C \\ {\scriptstyle f}\downarrow & \varphi & \downarrow{\scriptstyle g} \\ B & \underset{S}{\nrightarrow} & D \end{array}$	Cell exists iff $\forall (a,c) \in A \times C$, $(a,c) \in R \Rightarrow (f(a), g(c)) \in S$	Cell exists iff $\forall (a,c) \in A \times C$, $(f(a), g(c)) \in S \Rightarrow (a,c) \in R$
Loose arrow composition $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$	$R \otimes S := \{(a,c) \mid \exists b \in B.\ (a,b) \in R \wedge (b,c) \in S\}$	$R \otimes S := \{(a,c) \mid \forall b \in B.\ (a,b) \in R \vee (b,c) \in S\}$
Loose identities	$id^{\circ}_A := \{(a,a') \in A \times A \mid a = a'\}$	$id^{\bullet}_A := \{(a,a') \in A \times A \mid a \neq a'\}$

Companions + Conjoints

Definition (Grandis-Paré 2004) Let $f : A \to B$ be a tight arrow in a double category $\mathbb{D}$. A loose arrow $f_* : A \mathrel{\mapstochar\rightarrow} B$ is a companion of f if there exist cells α, β (below) such that $\beta \mid \alpha = 1_{f_*}$ and $\frac{\beta}{\alpha} = id_f$.

$$
\begin{tikzcd}
A \arrow[r, "f_*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, equals] \\
B \arrow[r, equals, "\shortmid"{marking}] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equals, "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\beta"] & A \arrow[d, "f"] \\
A \arrow[r, "f_*"', "\shortmid"{marking}] & B
\end{tikzcd}
$$

Definition (Grandis-Paré 2004) Let $f : A \to B$ be a tight arrow in a double category $\mathbb{D}$. A loose arrow $f^* : B \mathrel{\mapstochar\rightarrow} A$ is a conjoint of f if there exist cells η, ε (below) such that $\varepsilon \mid \eta = 1_{f^*}$ and $\frac{\eta}{\varepsilon} = id_f$.

$$
\begin{tikzcd}
A \arrow[r, equals, "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\eta"] & A \arrow[d, equals] \\
B \arrow[r, "f^*"', "\shortmid"{marking}] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "f^*", "\shortmid"{marking}] \arrow[d, equals] \arrow[dr, phantom, "\varepsilon"] & A \arrow[d, "f"] \\
B \arrow[r, equals, "\shortmid"{marking}] & B
\end{tikzcd}
$$

Equipments

Definition-Theorem (Shulman 2008) A double category $\mathbb{D}$ is an *equipment* if any of the following equivalent conditions hold:

(i) $\langle s,t \rangle : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is a fibration;

(ii) $\langle s,t \rangle : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is an opfibration;

(iii) $\mathbb{D}$ has all companions and conjoints.

Examples

- $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ (spans with one leg an identity)
- $\mathbb{P}\mathrm{rof}$ (representable profunctors)
- $\mathbb{R}\mathrm{el}^{\circ}$ (graphs/opgraphs of functions)
- $\mathbb{R}\mathrm{el}^{\bullet}$ (complements of graphs/opgraphs of functions)
- $\mathbb{Q}(\mathcal{K})$ has companions, and has conjoints iff every 1-cell is a map (i.e. has a right adjoint)

Elementary Existential Fibrations (EEFs)

Definition (Nasu 2025) A fibration $p : \mathcal{E} \to \mathcal{B}$ is said to be elementary existential if:

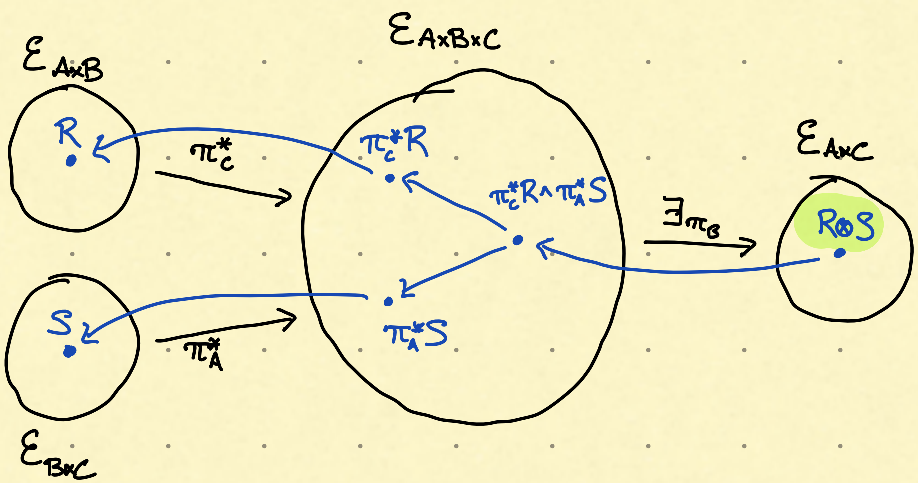
- $\mathcal{B}$ has finite products;
- $\mathcal{E}$ has fibred finite products (that are preserved by reindexing);
- For every parameterized diagonal $\Delta : X \times Y \to X \times Y \times Y$ in $\mathcal{B}$, the reindexing functor $\Delta^* : \mathcal{E}_{X \times Y \times Y} \to \mathcal{E}_{X \times Y}$ has a left adjoint $\exists_\Delta$ satisfying Beck-Chevalley and Frobenius reciprocity;
- For every projection $\pi_Y : X \times Y \to X$ in $\mathcal{B}$, the reindexing functor $\pi_Y^* : \mathcal{E}_X \to \mathcal{E}_{X \times Y}$ has a left adjoint $\exists_{\pi_Y}$ satisfying Beck-Chevalley and Frobenius reciprocity.

Remark The internal logic of elementary existential fibrations is regular logic (modulo some proof-theoretic considerations), i.e. the $\{\exists, \wedge, \top\}$ fragment of first-order logic.

$\mathbb{B}\mathrm{il}$ Construction

Definition (Nasu 2025) Let $p:\mathcal{E}\to\mathcal{B}$ be an elementary existential fibration (EEF). Then the following data form a (Frobenius cartesian) equipment $\mathbb{B}\mathrm{il}(p)$:

- the tight part of $\mathbb{B}\mathrm{il}(p)$ is the category $\mathcal{B}$;
- loose arrows $A \nrightarrow B$ in $\mathbb{B}\mathrm{il}(p)$ are objects of $\mathcal{E}$ over $A\times B$;
- cells $\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd}$ in $\mathbb{B}\mathrm{il}(p)$ are morphisms $\alpha: R\to S$ in $\mathcal{E}$ over $f\times g$;
- loose identity on A is given by $\exists_{\Delta_A}(\top_A)\in\mathcal{E}_{A\times A}$, when $\top_A$ is the terminal object of $\mathcal{E}_A$;
- given loose arrows $R: A\nrightarrow B$, $S: B\nrightarrow C$, the composite $R\otimes S$ is given as below:



$\mathbb{B}$il Construction (cont'd)

Proof Segment (Associators) Let $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$ be loose arrows in $\mathbb{B}\mathrm{il}(p)$.

$$
\begin{aligned}
\exists_{\pi_C}\left[\pi_D^* \exists_{\pi_B}(\pi_C^* R \wedge \pi_A^* S) \wedge \pi_A^* T\right] &\cong \exists_{\pi_C}\left[\exists_{\pi'_B} \pi'^*_D(\pi_C^* R \wedge \pi_A^* S) \wedge \pi_A^* T\right] && \text{(B-C)} \\
&\cong \exists_{\pi_C}\left[\exists_{\pi'_B}\left(\pi'^*_D(\pi_C^* R \wedge \pi_A^* S) \wedge \pi'^*_B \pi_A^* T\right)\right] && \text{(FR)} \\
&\cong \exists_{\pi_C}\left[\exists_{\pi'_B}\left((\pi'^*_D \pi_C^* R \wedge \pi'^*_D \pi_A^* S) \wedge \pi'^*_B \pi_A^* T\right)\right] && (\pi^* \text{ pres. } \vee) \\
&\;\;\vdots \\
&\cong \exists_{\pi_B}\left[\exists_{\pi'_C}\left(\pi'^*_C \pi_D^* R \wedge (\pi'^*_A \pi_D^* S \wedge \pi'^*_A \pi_B^* T)\right)\right] \\
&\cong \exists_{\pi_B}\left[\exists_{\pi'_C}\left(\pi'^*_C \pi_D^* R \wedge \pi'^*_A(\pi_D^* S \wedge \pi_B^* T)\right)\right] && (\pi^* \text{ pres. } \vee) \\
&\cong \exists_{\pi_B}\left[\pi_D^* R \wedge \exists_{\pi'_C} \pi'^*_A(\pi_D^* S \wedge \pi_B^* T)\right] && \text{(FR)} \\
&\cong \exists_{\pi_B}\left[\pi_D^* R \wedge \pi_A^* \exists_{\pi_C}(\pi_D^* S \wedge \pi_B^* T)\right] && \text{(B-C)}
\end{aligned}
$$

Connecting Fibrations and Double Categories

Prop. (Nasu 2025) There is a biequivalence

$$\begin{tikzcd} \mathrm{EEF} \arrow[r, bend left, "\mathbb{B}\mathrm{il}/\mathbb{F}\mathrm{r}"] \arrow[r, phantom, "\simeq"] & \mathrm{Eqp}_{\mathrm{Frob}} \arrow[l, bend left, "\mathrm{uni}"] \end{tikzcd}$$

between suitable 2-categories of EEFs and Frobenius cartesian equipments.

Examples

EEF p	$\mathbb{B}\mathrm{il}(p)$
Codomain fibration $\mathcal{B}^{\to} \to \mathcal{B}$ ($\mathcal{B}$ a category with finite limits)	$\mathbb{S}\mathrm{pan}(\mathcal{B})$
Subobject fibration $\mathrm{Pred} \to \mathrm{Set}$ (could use $\mathrm{Sub}(\mathcal{R}) \longrightarrow \mathcal{R}$ for $\mathcal{R}$ any regular category)	$\mathbb{R}\mathrm{el}^{\circ}$

Dual Fibrations

Definition: Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with corresponding indexed category $p^{-1} : \mathcal{B} \to \mathcal{C}at$. The dual fibration $p^{\circ} : \mathcal{E}^{\circ} \to \mathcal{B}$ of p is the fibration associated with the composite

$$\mathcal{B} \xrightarrow{p^{-1}} \mathcal{C}at \xrightarrow{(-)^{op}} \mathcal{C}at^{co}.$$

In elementary terms, $\mathcal{E}^{\circ}$ is a category with the same objects as $\mathcal{E}$, and morphisms (equivalence classes of) spans $L \xleftarrow{v} X \xrightarrow{h} R$ where v is vertical and h is cartesian.

Prop. (Kock 2015) There is a canonical isomorphism $\mathcal{E} \cong \mathcal{E}^{\circ\circ}$ over $\mathcal{B}$.

Remark: Taking the dual fibration "flips" the direction of adjunctions between fibres.

Formalizing (Categorical) Duality

Defn (Shulman 2016) A *strong duality involution* on a strict 2-category $\mathcal{A}$ consists of:

- a strict 2-functor $(-)^\bullet : \mathcal{A}^{co} \longrightarrow \mathcal{A}$;
- a strict 2-natural isomorphism η

$$\begin{tikzcd} \mathcal{A} \arrow[rr, equal] \arrow[dr, "((-)^\bullet)^{co}"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{A} \\ & \mathcal{A}^{co} \arrow[ur, "(-)^\bullet"'] & \end{tikzcd}$$

- an identity modification ζ

$$\begin{tikzcd} \mathcal{A}^{co} \arrow[r, "(-)^\bullet"] & \mathcal{A} \arrow[rr, equal] \arrow[dr, "((-)^\bullet)^{co}"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{A} \\ & & \mathcal{A}^{co} \arrow[ur, "(-)^\bullet"'] & \end{tikzcd} \overset{\zeta}{\Longrightarrow} \begin{tikzcd} \mathcal{A}^{co} \arrow[rr, equal] \arrow[dr, "(-)^\bullet"'] & \arrow[d, phantom, "\Downarrow\eta^{co}"] & \mathcal{A}^{co} \arrow[r, "(-)^\bullet"] & \mathcal{A} \\ & \mathcal{A} \arrow[ur, "((-)^\bullet)^{co}"'] & & \end{tikzcd}$$

whose components are therefore identity 2-cells $\zeta_x : \eta_{x^\bullet} \overset{=}{\Rightarrow} (\eta_x)^\bullet$.

Formalizing (Categorical) Duality (cont'd)

Examples

(i) "With nearly any reasonable set-theoretic definitions of 'category' and 'opposite'," the 2-category Cat of categories and functors has a strong (indeed strict) duality involution.

(ii) The 2-category Fib_B of fibrations over a fixed base category B has a strong duality involution sending fibrations to their duals.

A Notion of Self-Duality

Definition Let S be a sub-2-category of Fib_B. We say that S is *conceptually self-dual* if the duality involution on Fib_B restricts to a duality involution on S.

Examples Assume B has finite limits throughout.

(i) The sub-2-category of Boolean hyperdoctrines and appropriately structure-preserving morphisms is conceptually self-dual.

(ii) The analogous sub-2-category of EEFs is not conceptually self-dual, since the dual fibration of an EEF is not an EEF.

(iii) Fib_B itself is conceptually self-dual.

Since EEFs are not self-dual, $\mathbb{B}\mathrm{il}(p^\circ)$ is not definable for p an EEF.

So what are the duals to EEFs?

A Swath of Dual Definitions

Definition Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration. We say that p is *coprimary* if:

(i) $\mathcal{B}$ has finite products;

(ii) $\mathcal{E}$ has fibred finite coproducts.

Definition Let $p : \mathcal{E} \to \mathcal{B}$ be a coprimary fibration. We say that p is *universal* if, for every projection $\pi_Y : X \times Y \to X$ in $\mathcal{B}$, the reindexing functor $\pi_Y^* : \mathcal{E}_X \to \mathcal{E}_{X \times Y}$ has a right adjoint $\forall_{\pi_Y}$ satisfying (dual) Beck-Chevalley and (dual) Frobenius reciprocity.

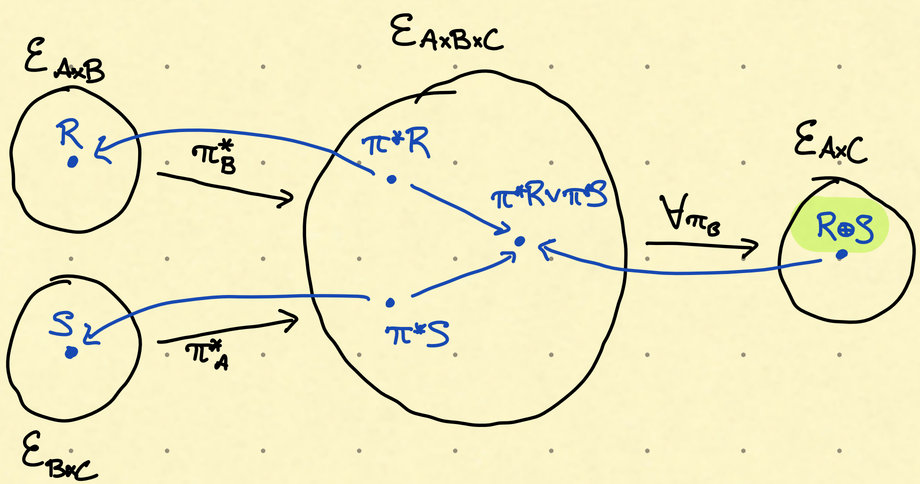
Definition Let $p : \mathcal{E} \to \mathcal{B}$ be a coprimary fibration. We say that p is *distinct* if, for every parameterized diagonal $\Delta : X \times Y \to X \times Y \times Y$ in $\mathcal{B}$, the reindexing functor $\Delta^* : \mathcal{E}_{X \times Y \times Y} \to \mathcal{E}_{X \times Y}$ has a right adjoint $\forall_\Delta$ satisfying (dual) Beck-Chevalley and (dual) Frobenius reciprocity.

Upshot p distinct universal fibration $\iff$ p° elementary existential fibration

A Dual $\mathbb{B}\mathrm{il}$ Construction

Proposition (M.) Let $p : \mathcal{E} \to \mathbb{B}$ be a distinct universal fibration. The following data form a double category $\mathbb{B}\mathrm{il}^{\circ}(p)$:

- the tight part of $\mathbb{B}\mathrm{il}^{\circ}(p)$ is $\mathbb{B}$;
- loose arrows $A \nrightarrow B$ are objects in the fibre $\mathcal{E}_{A \times B}$;
- cells
$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & D \end{tikzcd}$$
in $\mathbb{B}\mathrm{il}^{\circ}(p)$ are given by morphisms $\alpha : R \to S$ in $\mathcal{E}$ over $f \times g$;
- loose identity on A is given by $\forall_{\Delta_A}(\perp_A) \in \mathcal{E}_{A \times A}$, where $\perp_A$ is the initial object in $\mathcal{E}_A$;
- given loose arrows $R : A \nrightarrow B$, $S : B \nrightarrow C$, the composite $R \oplus S$ is given as below:



A Dual $\mathbb{B}$il Construction (cont'd)

Proof Segment (Unitors) Let $R : A \nrightarrow B$ be a loose arrow in $\mathbb{B}\mathrm{il}^{\circ}(p)$.

$$\forall_{\pi_A}\left(\pi_B^* \forall_{\Delta_A}(\bot_A) \vee \pi_A^* R\right) \cong \forall_{\pi_A}\left(\forall_{\Delta} \pi_B^* (\bot_A) \vee \pi_A^* R\right) \quad \text{(B-C)}$$

$$\cong \forall_{\pi_A}\left(\forall_{\Delta}(\bot_{A\times B}) \vee \pi_A^* R\right) \quad (\pi^* \text{ pres. } \bot)$$

$$\cong \forall_{\pi_A}\left(\forall_{\Delta}\left(\bot_{A\times B} \vee \Delta^* \pi_A^* R\right)\right) \quad \text{(FR)}$$

$$\cong \forall_{\pi_A}\left(\forall_{\Delta}\left(\Delta^* \pi_A^* R\right)\right)$$

$$\cong \forall_{1_{A\times B}}\left(1_{A\times B}^* R\right)$$

$$\cong R$$

A Dual $\mathbb{B}\mathrm{il}$ Construction (cont'd)

Lemma (M.) Let p be a distinct universal fibration. Then

$$\mathbb{B}\mathrm{il}^\bullet(p) \simeq \mathbb{B}\mathrm{il}(p^\bullet)$$

as double categories.

Conjecture (M.) Let $\mathrm{Fib}_{EE,DU}$ denote the sub-2-category of $\mathrm{Fib}_{\mathbb{B}}$ (for fixed $\mathbb{B}$ with finite products) consisting of those fibrations which are both elementary existential and distinct universal (with suitable morphisms). Then $\mathrm{Fib}_{EE,DU}$ is conceptually self-dual.

Conjecture (M.) For any sub-2-category S of $\mathrm{Fib}_{\mathbb{B}}$, the (2-)pullback

$$\begin{tikzcd} \text{"}S \cap S^\bullet\text{"} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & S^\bullet \arrow[d, hook] \\ S \arrow[r, hook] & \mathrm{Fib}_{\mathbb{B}} \end{tikzcd}$$

is conceptually self-dual.

Future Work

- Characterize double categories (essentially) of the form $\mathbb{B}\mathrm{il}(p)$ for p elementary existential + distinct universal

- What is the internal logic of such fibrations?

- Observe that a cell α in an equipment can be equivalently written in 4 ways:

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid" marking] \arrow[d, equal] & C \arrow[r, "g_*", "\shortmid" marking] \arrow[d, phantom, "\alpha"] & D \arrow[d, equal] \\ A \arrow[r, "f_*"', "\shortmid" marking] & B \arrow[r, "S"', "\shortmid" marking] & D \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f^*", "\shortmid" marking] \arrow[d, equal] & A \arrow[r, "R", "\shortmid" marking] \arrow[d, phantom, "\alpha'"] & C \arrow[d, equal] \\ B \arrow[r, "S"', "\shortmid" marking] & D \arrow[r, "g^*"', "\shortmid" marking] & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rrr, "R", "\shortmid" marking] \arrow[d, equal] & & & C \arrow[d, equal] \\ A \arrow[r, "f_*"', "\shortmid" marking] & B \arrow[r, "S"', "\shortmid" marking] \arrow[u, phantom, "\alpha^r"] & D \arrow[r, "g^*"', "\shortmid" marking] & C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f^*", "\shortmid" marking] \arrow[d, equal] & A \arrow[r, "R", "\shortmid" marking] & C \arrow[r, "g_*", "\shortmid" marking] \arrow[d, phantom, "\alpha^l"] & D \arrow[d, equal] \\ B \arrow[rrr, "S"', "\shortmid" marking] & & & D \end{tikzcd}$$

and that cells in $\mathbb{R}\mathrm{el}^{\bullet}$ are precisely cells in $\mathbb{R}\mathrm{el}^{\circ}$ of the form $(\alpha^r)^{op}$. What notions of duality can we get by taking duals of the other three kinds of cells?

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Thank you!

uni Construction

Definition (Nasu 2025) Let $\mathbb{D}$ be a cartesian equipment. Define $\mathrm{uni}(\mathbb{D})$ to be the fibration obtained from the pullback

$$\begin{tikzcd}
\mathrm{Uni}(\mathbb{D}) \arrow[r] \arrow[d, "\mathrm{uni}(\mathbb{D})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D}_1 \arrow[d, "\langle s, t \rangle"] \\
\mathbb{D}_0 \arrow[r, "A \mapsto (A, *)"'] & \mathbb{D}_0 \times \mathbb{D}_0
\end{tikzcd}$$

Lemma (Nasu 2025) Let $\mathbb{D}$ be a Frobenius cartesian equipment. Then $\mathbb{B}\mathrm{il}(\mathrm{uni}(\mathbb{D})) \simeq \mathbb{D}$.

Proof sketch Frobenius condition allows 'transfer of variables'; in particular, every loose arrow $R : A \nrightarrow B$ is isomorphic to a loose arrow $R' : A \times B \nrightarrow *$, so that pulling back to $\mathrm{uni}(\mathbb{D})$ does not "lose information".

Theorem (Nasu 2025) There is a biequivalence between appropriate 2-categories of EEFs and Frobenius cartesian equipments.

Retrocells

Definition (Paré 2024) Let $\mathbb{D}$ be a double category with chosen companions. A retrocell in $\mathbb{D}$ (left) corresponds to a (standard) double cell of the form (right) in $\mathbb{D}$:

$$
\begin{tikzcd}
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha \Nearrow"] & C \arrow[d, "g"] \\
B \arrow[r, "S"', "\shortmid"{marking}] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "f_*", "\shortmid"{marking}] \arrow[d, equal] & B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & D \arrow[d, equal] \\
A \arrow[r, "R"', "\shortmid"{marking}] & C \arrow[r, "g_*"', "\shortmid"{marking}] & D
\end{tikzcd}
$$

Theorem (Paré) The objects, tight and loose arrows of $\mathbb{D}$, together with retrocells, form a double category $\mathbb{D}^{\mathrm{ret}}$.

Theorem (Paré) (1) $\mathbb{D}^{\mathrm{ret}}$ has a canonical choice of companions (namely, the companions of $\mathbb{D}$).

(2) There is a canonical isomorphism of double categories with companions

$$\mathbb{D} \xrightarrow{\cong} \mathbb{D}^{\mathrm{ret\,ret}}$$

which is the identity on objects and tight and loose arrows.

Coretrocells

<u>Definition (Paré 2024)</u> Let $\mathbb{D}$ be a double category with chosen conjoints. A coretrocell in $\mathbb{D}$ (left) is a cell of the form (right) in $\mathbb{D}$:

$$\begin{tikzcd} A \arrow[r, "R", "+"{marking}] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "S"', "+"{marking}] \arrow[ur, phantom, "\Nwarrow\alpha"] & D \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "+"{marking}] \arrow[d, equal] & D \arrow[r, "g^*", "+"{marking}] \arrow[d, phantom, "\alpha"] & C \arrow[d, equal] \\ B \arrow[r, "f^*"', "+"{marking}] & A \arrow[r, "R"', "+"{marking}] & C \end{tikzcd}$$

<u>Theorem (Paré)</u> (1) The objects, tight and loose arrows of $\mathbb{D}$, together with coretrocells, form a double category $\mathbb{D}^{crt}$.

(2) $\mathbb{D}^{crt}$ has a canonical choice of conjoints.

(3) There is a canonical isomorphism of double categories with conjoints

$$\mathbb{D} \xrightarrow{\cong} \mathbb{D}^{crt\ crt}$$

which is the identity on objects and tight and loose arrows.

Considering Both Sides: Two-Sided Fibrations

Definition (Street) A span $A \xleftarrow{q} \mathcal{E} \xrightarrow{p} B$ in a representable 2-category $\mathcal{K}$ is a two-sided fibration if it is a (Φ_A, Φ_B)-bimodule, where Φ_A is the comma object of the identity cospan on A, seen as a pseudomonad in $\mathrm{Span}(\mathcal{K})$.

In Cat, this means:

(i) every morphism $f : a \to px$ in B has a p-cartesian lift $f^*a \to x$ that is q-vertical;

(ii) every morphism $g : qx \to b$ in A has a q-opcartesian lift $x \to g_*b$ that is p-vertical;

(iii) for every cartesian-opcartesian composite $f^*a \longrightarrow x \longrightarrow g_*b$, the canonical morphism $g_*f^*a \longrightarrow f^*g_*b$ is an isomorphism.

A span $A \xleftarrow{q} \mathcal{E} \xrightarrow{p} B$ is a two-sided opfibration if the opposite span $B \xleftarrow{p} \mathcal{E} \xrightarrow{q} A$ is a two-sided fibration.

Two-Sided Fibrations (cont'd)

Remark Given a two-sided fibration $A \xleftarrow{q} \mathcal{E} \xrightarrow{p} B$, every morphism $I \to J$ in $\mathcal{E}$ factors into three:

$$I \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} J$$

where α is q-opcartesian and p-vertical, β is p,q-vertical, and γ is p-cartesian and q-vertical.

Definition (von Glehn) Let $A \xleftarrow{q} \mathcal{E} \xrightarrow{p} B$ be a two-sided fibration with corresponding indexed category $(p,q)^{-1} : A^{op} \times B \to \mathcal{C}at$. Its dual two-sided fibration $A \xleftarrow{q^\circ} \mathcal{E}^\circ \xrightarrow{p^\circ} B$ is the two-sided fibration associated with the composite $A^{op} \times B \xrightarrow{(q,p)^{-1}} \mathcal{C}at \xrightarrow{(-)^{op}} \mathcal{C}at^{co}$.

In elementary terms, $\mathcal{E}^\circ$ has the same objects as $\mathcal{E}$, and morphisms (equivalence classes of) diagrams

$$I \xrightarrow{\alpha} \bullet \xleftarrow{\beta^\circ} \bullet \xrightarrow{\gamma} J$$

where α is q-opcartesian and p-vertical, β° is p,q-vertical, and γ is p-cartesian and q-vertical.

Two-Sided Fibrations (cont'd)

Lemma (Capucci) A double category has companions (resp. conjoints) iff its underlying source-target span is a two-sided opfibration (resp. fibration).

Lemma (M.) Let $\mathbb{D}$ be a double category with chosen companions (resp. conjoints). The underlying source-target span of $\mathbb{D}^{\mathrm{ret}}$ (resp. $\mathbb{D}^{\mathrm{crt}}$) is (isomorphic to) the dual two-sided fibration of that of $\mathbb{D}$.

Proof Sketch Seeing cells in $\mathbb{D}$ as morphisms in the total category of a two-sided opfibration, every cell can be factored as on the right. Taking the dual two-sided fibration amounts to flipping the middle cells, which become precisely retrocells.

$$\begin{tikzcd}
A \arrow[rr, "R", "\shortmid"{marking}] \arrow[d, equal] & \arrow[d, phantom, "\Downarrow"] & C \arrow[d, "g"] \\
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, equal] & C \arrow[r, "g_*", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & D \arrow[d, equal] \\
A \arrow[r, "f_*", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & D \arrow[d, equal] \\
B \arrow[rr, "S"', "\shortmid"{marking}] & & D
\end{tikzcd}$$