Examples of Arrow Algebras

Benno van den Berg, Max Wehmeier

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Overview

1. Arrow Algebras
   - Motivation: Locales
   - Definition
2. The Herbrand Arrow Algebra
   - Definition
   - Additional Properties
3. The Diller-Nahm Dialectica Arrow Algebra
   - Definition
   - Additional Properties

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Locales - Definition and Examples

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A *Locale* is a complete poset $(L, \preccurlyeq)$ satisfying the *infinite distributivity law*: For all $a \in L,\ B \subseteq L$, we have

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  Conversely, define a Heyting-Implication on a Locale:

$$a \to b := \bigcurlyvee \{c : c \curlywedge a \preccurlyeq b\}.$$

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Tripos (Hyland, Johnstone, Pitts)

By $\mathbf{preHeyt}$ we denote the category of *preHeyting algebras*, i.e. preorders, whose poset reflection is a Heyting algebra. A *tripos* is a pseudofunctor $\mathsf{P} \colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{preHeyt}$, such that

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- for each function $f\colon Y \to X$, the image $\mathsf{P}f\colon \mathsf{P}X \to \mathsf{P}Y$ has both adjoints in $\mathbf{Preord}$ satisfying the Beck-Chevalley condition and
- there is a set Prop and an element $\top \in \mathsf{P}(\mathrm{Prop})$, such that for all $\phi \in \mathsf{P}X$, there is a map $[\phi]\colon X \to \mathrm{Prop}$, such that $\mathsf{P}([\phi])(\top) \cong \phi$.

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Example

For a locale $(L, \preccurlyeq)$, $\mathsf{P}_L(X) := L^X$ forms a tripos with the pointwise order.

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An *arrow structure* is a complete poset $(A, \preccurlyeq)$ with a binary operation $\to : A \times A \to A$, such that whenever $a' \preccurlyeq a$ and $b \preccurlyeq b'$, then

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- Let $m \cdot n = \varphi_m(n)$ be *Kleene application*, i.e. the result of the m-th partial recursive function on input n, whenever defined. Then $(\mathrm{Pow}(\mathbb{N}), \subseteq)$ with

$$X \to Y := \{e : \text{For all } x \in X, \text{ we have that } e \cdot x \text{ exists and } e \cdot x \in Y\},$$

carries an arrow structure.

Separator

If we have an arrow structure $(A, \preccurlyeq, \to)$, we sometimes want to consider more elements than just $\top$ to be true. For this, we use the separator. It is a designated subset of A of values that we consider to be true.

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contains the combinators: $\mathbf{k}, \mathbf{s}, \mathbf{a} \in S$ for

$$\mathbf{k} := \bigcurlywedge_{a,b \in A} a \rightarrow b \rightarrow a,$$

$$\mathbf{s} := \bigcurlywedge_{a,b,c \in A} (a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow (a \rightarrow c),$$

$$\mathbf{a} := \bigcurlywedge_{a \in A, B \subseteq \mathrm{Im}(\rightarrow)} (\bigcurlywedge_{b \in B} a \rightarrow b) \rightarrow a \rightarrow (\bigcurlywedge_{b \in B} b).$$

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Logic in an Arrow Algebra

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Proposition ([BB25, Prop. 3.5])

Let $(A, \preccurlyeq, \to, S)$ be an Arrow Algebra. Suppose φ is an propositional formula with free variables $p_1, \ldots, p_n$ in the implicative fragment. If φ is an intuitionistic tautology, we get

$$\bigcurlywedge_{p_1,\ldots,p_n \in A} \varphi \in S.$$

The Arrow Tripos

Given an Arrow Algebra $\mathcal{A} = (A, \preccurlyeq, \to, S)$, we can define a mapping $\mathbf{Set}^{\mathrm{op}} \to \mathbf{preHeyt}$ by sending a set X to $\mathsf{P}_{\mathcal{A}}(X) := (A^X, \vdash_{A^X})$, where

$$\varphi \vdash_{A^X} \psi \quad :\Leftrightarrow \quad \bigcurlywedge_{x \in X} \varphi(x) \to \psi(x) \in S.$$

We call $\vdash_{A^X}$ the *logical order*.

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Theorem ([BB25, Thm. 5.10])

For an Arrow Algebra $\mathcal{A} = (A, \preccurlyeq, \to, S)$, the mapping $\mathsf{P}_{\mathcal{A}}$ is a tripos, which we call the *Arrow Tripos*.

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This recovers all localic triposes as well as the effective tripos and various other realizability triposes.

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$$\Sigma = \{(A_0, A_1) : A_0, A_1 \subseteq \mathcal{P}, A_0 \subseteq P_f(A_1), A_0 \text{ is cl. upwards in } P_f(A_1)\}.$$

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Implication is given by $(C_0, C_1) = (A_0, A_1) \to (B_0, B_1)$, where

$$C_1 := \{c \in \mathcal{P} : (\forall m \in P_f(A_1))\ c \cdot m\downarrow \text{ and } c \cdot m \in P_f(B_1)\},$$
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Proposition (Herbrand Arrow Algebra)

The quadruple $(\Sigma, \preccurlyeq, \to, S)$ is an Arrow Algebra.

Additional Properties

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- *strong* if

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- *an Implicative Algebra* if for all $a \in A$, $B \subseteq A$, we have
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- *an Implicative Algebra* if for all $a \in A$, $B \subseteq A$, we have
$$a \to \bigcurlywedge_{b \in B} b = \bigcurlywedge_{b \in B} a \to b$$
and a *Quasi-Implicative Algebra* if this holds for all $B \neq \emptyset$ [Miq20].

Relation of the additional properties

$$\begin{tikzcd}
& \text{Implicative Algebra} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{Quasi-Implicative Arrow Algebra} \arrow[d, Rightarrow] & & \text{Strong Arrow Algebra} \arrow[ddl, Rightarrow] \\
\text{Binary Implicative Arrow Algebra} \arrow[dr, Rightarrow] & & \\
& \text{Arrow Algebra} &
\end{tikzcd}$$

Additional Properties of the Herbrand Arrow Algebra

Proposition

The Herbrand Arrow Algebra satisfies none of the additional properties.

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If the Herbrand Arrow Algebra was strong, i.e. **a'** $\in S$, then the Halting problem would be decidable.

By upwards closure, we would have $\top \to (\top \to \top) \in S$ and there would be an element $t \in (\top \to (\top \to \top))_0$, which we could use to make any element in $\top$, especially any partial function, to an element of $\top \to \top$, i.e. a total function, which extends it.

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For a partial combinatory algebra $\mathcal{P}$, let

$$\Sigma_{\mathrm{DN}} := \{(X, Y, R) \in \mathrm{Pow}(\mathcal{P}) \times \mathrm{Pow}(\mathcal{P}) \times \mathrm{Pow}(\mathcal{P} \times \mathcal{P}) : R \subseteq X \times Y\}.$$

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For $p = (X, Y, R) \in \Sigma_{\mathrm{DN}}$, we write $p^+ = X,\ p^- = Y,\ p(a, b) \Leftrightarrow R(a, b)$.

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The Dialectica interpretation is a translation from intuitionistic arithmetic to Σ^0_2-formulas, i.e. formulas of the form $\exists\overline{x}.\forall\overline{y}.A$, where A is quantifier-free.

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The Diller-Nahm Dialectica Tripos [Str]

For a partial combinatory algebra $\mathcal{P}$, let

$$\Sigma_{\mathrm{DN}} := \{(X, Y, R) \in \mathrm{Pow}(\mathcal{P}) \times \mathrm{Pow}(\mathcal{P}) \times \mathrm{Pow}(\mathcal{P} \times \mathcal{P}) : R \subseteq X \times Y\}.$$

Given a set X, we send it to ${\Sigma_{\mathrm{DN}}}^X$ with some order.

For $p = (X, Y, R) \in \Sigma_{\mathrm{DN}}$, we write $p^+ = X,\ p^- = Y,\ p(a, b) \Leftrightarrow R(a, b)$.

The Dialectica interpretation is a translation from intuitionistic arithmetic to Σ^0_2-formulas, i.e. formulas of the form $\exists \overline{x}. \forall \overline{y}. A$, where A is quantifier-free. Then the Diller-Nahm Variant can be understood via game semantics, where first arguments in favour have to be chosen, then arguments against and finally we check which arguments "defeat" which.

The Diller-Nahm Dialectica Arrow Algebra

Implication in the Diller-Nahm Dialectica Tripos is given by

$$\begin{aligned}(p \to q)^+ &:= (p^+ \to q^+) \times (p^+ \times q^- \to P_f(p^-)) \\ (p \to q)^- &:= p^+ \times q^- \\ (p \to q)(\langle e^+; e^- \rangle, \langle a; b \rangle) &:\Leftrightarrow ([\forall c \in e^- \langle a; b \rangle . p(a, c)] \Rightarrow q(e^+ a, b))\end{aligned}$$

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We can also define an order by letting

$$p \preccurlyeq q \;\; :\Leftrightarrow \;\; p^+ \subseteq q^+, \;\; p^- \supseteq q^-, \;\; \forall a \in p^+, b \in q^- . p(a, b) \Rightarrow q(a, b).$$

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Proposition (Diller-Nahm Dialectica Arrow Algebra)

The quadruple $(\Sigma_{\mathrm{DN}}, \preccurlyeq, \to, S)$ is an Arrow Algebra.

Additional Properties of the Diller-Nahm Dialectica Arrow Algebra

Proposition

The Diller-Nahm Dialectica Arrow Algebra is strong and Quasi-Implicative, but not Implicative.

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The trick of using the Halting problem to refute strongness does not work in this case. It relies on the case where $B = \emptyset$ because since $\curlywedge \emptyset = \top$, we get a witness for $\top \to \top \to \top \in S$.

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$$= \top^+ \times a^+ \times \top^- = \emptyset.$$

So the special case is not applicable.

Conclusion

- We showed that Examples of Arrow Algebras are not hard to find. In many cases, the first attempt at defining an Arrow Algebra is the correct one.

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- We showed that Examples of Arrow Algebras are not hard to find. In many cases, the first attempt at defining an Arrow Algebra is the correct one.
- We extended Thomas Streicher's construction of the Diller-Nahm Dialectica interpretation to arbitrary partial combinatory algebras.
- Since the Herbrand Arrow Algebra is an Arrow Algebra, but satisfies no additional properties, this gives us confidence in the comparably weak axioms.

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The End

Thank you for your attention!

Are there questions?

The Gleason cover for Arrow Algebras

Gleason showed that every compact Hausdorff space has a *Gleason cover*, i.e. an extremally disconnected cover [Gle58]. Johnstone generalized this to Topoi, where the covering spaces are DeMorgan-Topoi [Joh80; Joh81]. The proof used the result

$$\mathrm{Loc}(\mathrm{Sh}(X)) \simeq \mathrm{Loc}\,/X,$$

which is due to Joyal and Tierney [JT84]. If we generalize this to Arrow Algebras, we hopefully can use a similar proof strategy to find a Gleason cover for Arrow Algebras. But already defining the internalization functor $\mathcal{I}\colon \mathrm{Arr}\,/\mathcal{A} \to \mathrm{Arr}(\mathrm{AT}(\mathcal{A}))$ is difficult to construct.

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