A duality theorem for constructive K

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Intuitionistic modal logic WK

Language: $\mathcal{L} \ni \varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi$

$$\Diamond\bot \to \bot \qquad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \qquad \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi) \qquad \frac{\varphi}{\Box\varphi}$$

WK-algebras: Heyting algebras with operators $\mathcal{H} = (H, \Box, \Diamond)$

WK-frames: $(W, \leq, R)$ with valuation $V : \mathrm{Prop} \to up(W, \leq)$

$$\begin{array}{lll} w \Vdash \varphi \to \psi & \text{iff} & \forall v \geq w \, (v \Vdash \varphi \text{ implies } v \Vdash \psi) \\ w \Vdash \Box\varphi & \text{iff} & \forall v \geq w \, (\forall u \, (vRu \text{ implies } u \Vdash \varphi)) \\ w \Vdash \Diamond\varphi & \text{iff} & \forall v \geq w \, (\exists u \, (vRu \text{ and } u \Vdash \varphi)) \end{array}$$

Example of a WK-frame

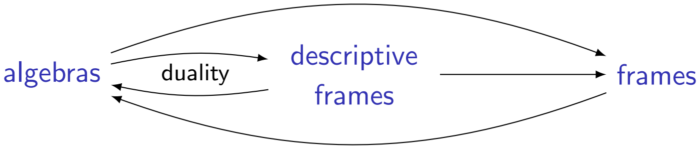
$$
\begin{tikzcd}
u \arrow[r, "R"] & s \Vdash q \\
w \arrow[u, "\leq"] \arrow[r, "R"'] & v \Vdash p
\end{tikzcd}
$$

$$w \Vdash \Box\varphi \text{ iff } \forall x \geq w \, (\forall y \, (xRy \text{ implies } y \Vdash \varphi))$$

$$w \Vdash \Diamond\varphi \text{ iff } \forall x \geq w \, (\exists y \, (xRy \text{ and } y \Vdash \varphi))$$

$$
\left.
\begin{array}{l}
\blacktriangleright \;\; w \Vdash \Diamond(p \vee q) \\
\blacktriangleright \;\; w \nVdash \Diamond q \\
\blacktriangleright \;\; w \nVdash \Diamond p
\end{array}
\right\} \; w \nVdash \Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)
$$

Why duality?



- Completeness theorems
- Sahlqvist style correspondence and canonicity
- Bisimilarity-somewhere-else
- Goldblatt-Thomason style theorems

Esakia duality for intuitionistic logic

A general (intuitionistic Kripke) frame is a tuple $(\underbrace{W, \leq}, A)$ where

intuitionistic Kripke frame $\leftarrow$ $W, \leq$

A $\rightarrow$ subalgebra of $up(W, \leq)$

- A contains $X, \emptyset$
- A is closed under $\cap, \cup, \rightarrow$

$$\mathfrak{G} = (W, \leq, A) \rightsquigarrow \mathfrak{G}^* = (A, X, \emptyset, \cap, \cup, \rightarrow)$$

$$\mathcal{H}_* = (pf(\mathcal{H}), \subseteq, \theta(\mathcal{H})) \leftsquigarrow \mathcal{H} = (H, \top, \bot, \wedge, \vee, \rightarrow)$$

- $\theta(a) = \{p \in pf(\mathcal{H}) \mid a \in p\}$
- $\theta(\mathcal{H}) = \{\theta(a) \mid a \in H\}$

- $\theta : \mathcal{H} \to (\mathcal{H}_*)^* : a \mapsto \{p \in pf(\mathcal{H}) \mid a \in p\}$ is a natural isomorphism
- When is $\eta : \mathfrak{G} \to (\mathfrak{G}^*)_* : x \mapsto \{a \in A \mid x \in a\}$ a natural isomorphism?

Esakia duality for intuitionistic logic

When is $\eta : \mathfrak{G} \to (\mathfrak{G}^*)_* : x \mapsto \{a \in A \mid x \in a\}$ a natural isomorphism?

- η is an embedding iff $\mathfrak{G}$ is refined, i.e.

$$x \not\leq y \quad \text{implies} \quad \exists a \in A \text{ s.t. } x \in a \text{ and } y \notin a$$

- η is surjective iff $\mathfrak{G}$ is compact, i.e.

$$\text{if } B \subseteq A \cup -A \text{ has the f.i.p. then } \bigcap B \neq \emptyset$$

- $\mathfrak{G}$ is descriptive if it is refined and compact

Esakia duality: $\mathbf{HA} \equiv^{\mathrm{op}} \mathbf{DF}$

Duality for WK

Adding modalities

$$\begin{tikzcd} \text{Heyting algebras} \arrow[r, bend left] \arrow[r, phantom, "\equiv^{\mathrm{op}}"] & \text{descriptive frames} \arrow[l, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \begin{array}{c}\text{Heyting algebra}\\ \text{with operators}\end{array} \arrow[r, bend left] \arrow[r, phantom, "\equiv^{\mathrm{op}}"] & \begin{array}{c}\text{descriptive frames}\\ \text{with extra structure}\end{array} \arrow[l, bend left] \end{tikzcd}$$

This gives dualities for

- Intuitionistic K (IK)
- Intuitionistic logic with one monotone modality
- Intuitionistic conditional logic

From WK-frames to WK-algebras

A general WK-frame is a tuple $\mathfrak{G} = (W, \leq, R, A)$ such that

- $(W, \leq, A)$ is a general frame
- if $a \in A$ then

$$\Box a := \{x \in W \mid \forall y \geq x\,(\forall z\,(yRz \Rightarrow z \in a))\} \in A$$
$$\Diamond a := \{x \in W \mid \forall y \geq x\,(\exists z\,(yRz \text{ and } z \in a))\} \in A$$

Then $\mathfrak{G}^* = (A, \Box, \Diamond)$ is a WK-algebra

Example

$$\begin{tikzcd}
v \arrow[r, "R"] \arrow[d, bend right, "\leq"'] & u \\
w \arrow[u, bend right, "\leq"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \{w, v, u\} \arrow[loop above, "\Diamond"] & \\
\{w, v\} \arrow[ur] \arrow[rr, bend right, "\Box"'] & & \{u\} \arrow[ul] \arrow[ul, bend right, "\Box"'] \\
 & \emptyset \arrow[ul] \arrow[ur] \arrow[ur, bend right, "\Box"'] &
\end{tikzcd}$$

$$\Box(\{u\}) = \{w, v, u\}$$
$$\Box(\{x, y\}) = \Box(\emptyset) = \{z\}$$
$$\Diamond A = \emptyset \text{ for all } A$$

From WK-algebras to WK-frames

Given A WK-algebra $\mathcal{H} = (H, \Box, \Diamond)$, let

$$\mathcal{H}_+ = (pf(H), \subseteq, R_{\mathcal{H}}, \theta(\mathcal{H}))$$

How should we define $R_{\mathcal{H}}$?

Guided by classical modal logic, let $pR_{\mathcal{H}}q$ iff for all $a \in H$:

$$\begin{array}{rcl} \Box a \in p & \text{implies} & a \in q \\ a \in q & \text{implies} & \Diamond a \in p \end{array}$$

$$\begin{array}{llll} \text{Then } \Diamond(a \vee b) \in p & \text{implies} & a \vee b \in q & (\text{for some } q \text{ with } pR_{\mathcal{H}}q) \\ & \text{implies} & a \in q \text{ or } b \in q & \\ & \text{implies} & \Diamond a \in p \text{ or } \Diamond b \in p & \\ & \text{implies} & \Diamond a \vee \Diamond b \in p & \end{array}$$

So $\mathcal{H}_+ \Vdash \Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$

Segments

A segment of a WK-algebra $\mathcal{H} = (H, \Box, \Diamond)$ is a pair (p, U) such that

prime filter ← p, U → set of prime filters

- $\Box a \in p$ implies $a \in q$ for all $q \in U$
- $\Diamond a \in p$ implies $a \in q$ for some $q \in U$

Define $\mathcal{H}_* = (seg(\mathcal{H}), \lesssim, R_{\mathcal{H}}, \theta(\mathcal{H}))$, where

$$\begin{aligned}(p, U) \lesssim (q, V) \quad &\text{iff} \quad p \subseteq q \\ (p, U)\; R_{\mathcal{H}}\; (q, V) \quad &\text{iff} \quad q \in U\end{aligned}$$

Then $\theta : \mathcal{H} \to (\mathcal{H}_*)^* : a \mapsto \{(p, U) \in seg(\mathcal{H}) \mid a \in p\}$ defines a natural iso

Segments

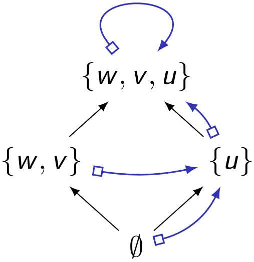
A segment of a WK-algebra $\mathcal{H} = (H, \Box, \Diamond)$ is a pair (p, U) such that

prime filter ⟵ p, U ⟶ set of prime filters

- $\Box a \in p$ implies $a \in q$ for all $q \in U$;
- $\Diamond a \in p$ implies $a \in q$ for some $q \in U$.

Example

$$\begin{tikzcd} v \arrow[r, "R"] \arrow[d, bend right, "\leq"'] & u \\ w \arrow[u, bend right, "\leq"'] & \end{tikzcd}$$



$$\begin{tikzcd} (p, \{q\}) \arrow[r, "R"] \arrow[d, bend right, "\leq"'] & (q, \emptyset) \\ (p, \emptyset) \arrow[u, bend right, "\leq"'] & \end{tikzcd}$$

- Prime filters: $p := \uparrow\{w, v\}$ and $q := \uparrow\{u\}$

Descriptive frames

Let $\mathfrak{G} = (X, \leq, R, A)$ be a general WK-frame, when is

$$\bar{\eta} : \mathfrak{G} \to (\mathfrak{G}^*)_* : x \mapsto (\eta(x), \{\eta(y) \mid xRy\})$$

a natural iso?

- if $\mathfrak{G}$ is refined then

$$x \leq y \quad \text{iff} \quad \bar{\eta}(x) \precsim \bar{\eta}(y)$$

- if additionally $\mathfrak{G}$ is modally defined, i.e. if xRy and $y \leq z \leq y$ then xRz, then

$$\bar{\eta} \text{ is an embedding}$$

Refinedness implies injectivity between clusters
Modal refinedness implies injectivity inside clusters

Duality for WK-algebras

Let $\mathfrak{G} = (X, \leq, R, A)$ be a general WK-frame, when is

$$\bar{\eta} : \mathfrak{G} \to (\mathfrak{G}^*)_* : x \mapsto (\eta(x), \{\eta(y) \mid xRy\})$$

a natural iso?

- if $\mathfrak{G}$ is compact then for every segment $(p, U) \in (\mathfrak{G}^*)_*$ there exists $x \in X$ such that $\eta(x) = p$
- finally, $\bar{\eta}$ is surjective if $\mathfrak{G}$ is compact and full, i.e. for all $x \in X$ and $U \in X$ s.t.
  - $x \in \Box a$ implies $U \subseteq a$
  - $x \in \Diamond a$ implies $U \cap a \neq \emptyset$

  there exists x' such that $x \leq x' \leq x$ and $R[x'] = U$
- $\mathfrak{G}$ is called descriptive if it is refined, modally refined, compact and full

Duality for WK: WK-Alg $\equiv^{\mathrm{op}}$ Descripctive WK-frames

Applications

Sahlqvist style correspondence

A Sahlqvist formula is an implication $\varphi \to \psi$ where

- φ is constructed from boxed atoms, $\top, \bot, \wedge$ and $\vee$
- ψ is constructed from $\top, \bot, \wedge, \vee, \Box, \Diamond$

Examples: $\Box p \to p, \quad p \to \Diamond p, \quad \Box p \to \Diamond p, \quad p \to \Box\Diamond p, \quad \Box\Box p \to \Box p$

Non-examples: $\Diamond p \to \Diamond\Diamond p, \quad \Diamond\Box p \to \Box\Diamond p$

Theorem: Every Sahlqvist formula has a computable, FO, global frame correspondent

- Classical first-order correspondence language with binary predicates $\leq$ and R
- Usual proof: standard translation and eliminating second order quantifiers
- Modification: $st_x(\Diamond\varphi) = \forall y((x \leq y) \to \exists z(yRz \wedge st_z(\varphi)))$
- $\forall\exists$-patterns complicates $\Diamond$ in antecedent

Sahlqvist style canonicity

Theorem: For any set Λ of Sahlqvist formulas, the logic $WK \oplus \Lambda$ is complete w.r.t. the class of frames characterised by the first-order frame correspondents generated from Λ

- Usual proof:

  validity in descriptive frame $\Rightarrow$ validity in underlying frame

- For WK:

  validity in descriptive frame (lots of segments)

  $\Rightarrow$ validity in related "semi-descriptive" frame (fewer segments)

  $\Rightarrow$ validity in underlying frame

The segment extension

The segment extension of a WK-frame $\mathfrak{X} = (X, \leq, R)$ is $\mathfrak{X}^{seg} := (\mathfrak{X}^+)_+$

$$\begin{tikzcd}
(X, \leq, R) \arrow[rr, bend left=10] \arrow[d, dashed, "\bar{\eta}"', "\text{not a bounded morphism!}"] & & (up(X, \leq), \Box, \Diamond) \arrow[dll, bend left=10] \\
(X^{seg}, \subsetneq, R^{seg}) & &
\end{tikzcd}$$

As before, $(p, U)R^{seg}(q, V)$ iff $q \in U$

Lemma: $\mathfrak{X}^{seg} \Vdash \varphi$ implies $\mathfrak{X} \Vdash \varphi$

Goldblatt-Thomason style theorem

We can define disjoint unions, generated subframes and bounded morphic images as expected

Define $Frm(\Phi) = \{\mathfrak{X} \mid \mathfrak{X}$ is a WK-frame and $\mathfrak{X} \Vdash \varphi$ for all $\varphi \in \Phi\}$

A class $\mathcal{K}$ of WK-frames is called axiomatic if $\mathcal{K} = Frm(\Phi)$ for some set Φ of formulas

Theorem: Let $\mathcal{K}$ be a class of WK-frames closed under segment extensions. TFAE:

- $\mathcal{K}$ is axiomatic
- $\mathcal{K}$ reflects segment extensions and is closed under disjoint unions, generated subframes and bounded morphic images

Conclusion

Observations

- Not based on Esakia duality
- Different variations based on choice of segments
- Many applications

Future work

- Extend class of Sahlqvist formulas
- Bisimulations, modal saturation and bisimilarity-somewhere-else
- Similar style duality for classical K

Thank you