Structural completeness in BL-algebras and Basic hoops

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Introduction

A large class of substructural logics is given by the finitary extensions of the **Full Lambek Calculus** $\mathcal{FL}$.
It is well known that all these finitary extensions are algebraizable with an equivalent algebraic semantics (in the sense of Blok-Pigozzi) that is at least a quasivariety of algebras.
In this context, one of the most studied examples is **Hájek Basic Logic**, which we denote by $\mathcal{BL}$, and its algebraic counterpart given by the variety of BL-**algebras**.

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In this context, one of the most studied examples is **Hájek Basic Logic**, which we denote by $\mathcal{BL}$, and its algebraic counterpart given by the variety of BL-**algebras**.

Since all extensions of $\mathcal{FL}$ have a primitive connective $\mathbf{0}$ that denotes the **falsum**, if $\mathcal{L}$ is an extension of $\mathcal{FL}$, it makes sense to study its **positive fragment** $\mathcal{L}^+$ (i.e. the logic obtained from $\mathcal{L}$ by removing $\mathbf{0}$ from the signature), which is still algebraizable.
In this presentation, we will analyze some properties of $\mathcal{BL}$ and its positive fragment $\mathcal{BL}^+$, via their equivalent algebraic semantics BL and BH, which are the varieties of **BL algebras** and **Basic hoops** respectively.

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A rule is **admissible** in a logic if, when added to its calculus, it does not produce new theorems.

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- A logic $\mathcal{L}$ is **structurally complete** if every admissible rule of $\mathcal{L}$ is derivable in $\mathcal{L}$.
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In an algebraizable logic, these two notions correspond to the associated quasivariety being **structural** and **primitive** respectively.

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In this work, we find some varieties of BL-algebras and basic hoops that are structural; moreover, we will show some classes of (quasi)varieties that are not.
In order to do so, we will rely on a particular sufficient condition for a (quasi)variety to be structural.

Preliminaries

We say that an algebra $\mathbf{A}$ is

- **projective** in a quasivariety Q if for all $\mathbf{B} \in \mathsf{Q}$, if $f : \mathbf{B} \longrightarrow \mathbf{A}$ is a surjective homomorphism, then there is an embedding $g : \mathbf{A} \longrightarrow \mathbf{B}$ with $gf = id_{\mathbf{A}}$;
- **weakly projective** if for all $\mathbf{B} \in \mathsf{Q}$, if $\mathbf{A} \in \mathbb{H}(\mathbf{B})$, then $\mathbf{A} \in \mathbb{IS}(\mathbf{B})$;
- **finitely presented** if there exist a finite set X and a compact congruence θ of $\mathbf{F}_{\mathsf{Q}}(X)$ such that $\mathbf{F}_{\mathsf{Q}}(X)/\theta \cong \mathbf{A}$;
- **unifiable** if there exists a homomorphism $f : \mathbf{A} \longrightarrow \mathbf{P}$, where $\mathbf{P}$ is projective in Q.

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The theorem that we will use is the following:

Theorem (Aglianò-Ugolini)

If a variety V has the Baker-Beynon property and every nontrivial finitely presented algebra in V is unifiable, then V is structural. If V is also locally finite, then it is primitive.

Varieties of BL-algebras and basic hoops

Definition

A **BL-algebra** is a commutative, integral and 0-bounded residuated lattice satisfying

1. divisibility: $x \wedge y = x \cdot (x \to y)$;
2. prelinearity: $(x \to y) \vee (y \to x) = 1$.

A **Wajsberg algebra** is a BL-algebra satisfying also involutivity, that is $\neg\neg x = x$.
Basic hoops and **Wajsberg hoops** are 0-free subreducts of BL-algebras and Wajsberg algebras respectively.

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All these classes are varieties, that we will denote by BL, WA, BH and WH respectively.

Ordinal sums

Notice that, since BL and BH are prelinear varieties, they are generated by chains, and so is every one of their subvarieties. Moreover, there is a deep connection between totally ordered BL-algebras (basic hoops) and totally ordered Wajsberg algebras (Wajsberg hoops), relying on the ordinal sum construction.

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Theorem (Aglianò-Montagna)

Every totally ordered basic hoop is a ordinal sum of a family of Wajsberg hoops; every totally ordered BL*-algebra is a ordinal sum of a family of Wajsberg hoops, the first of which is a Wajsberg algebra. Moreover since the sum of two nontrivial Wajsberg hoops is never a Wajsberg hoop the components of the sum are unique up to isomorphism.*

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For this reason from now on when we write a totally ordered basic hoop or BL-algebra as $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$, we always mean that the ordinal sum is composed of Wajsberg hoops, which will be called the **Wajsberg components** of $\mathbf{A}$.

Totally ordered Wajsberg hoops

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We define:

- the finite Wajsberg chain with $n+1$ elements $\mathbf{Ł}_n = \Gamma(\mathbb{Z}, n)$;
- for $0 \leq k \leq n$, the infinite Wajsberg chain $\mathbf{Ł}_{n,k} = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (n, k))$;
- the unbounded cancellative Wajsberg chain $\mathbf{C}_\omega$ that has as universe the free group on one generator, where the product is the group product and $a^l \to a^m = a^{\max(m-l,0)}$.

Usually the same symbol is used for Wajsberg chains regardless if they are Wajsberg algebras or Wajsberg hoops. However in our case this distinction becomes rather significant so we will denote the hoop reducts of $\mathbf{Ł}_n$ and $\mathbf{Ł}_{n,k}$ by $\mathbf{Ł}_n^+$ and $\mathbf{Ł}_{n,k}^+$ respectively.

Structural completeness in varieties of Basic hoops

Let us start our analysis from varieties of basic hoops.
Notice that in any variety V of hoops, the free algebra in V over 0 generators is the trivial algebra $\mathbf{T}$ with only one element. Since any free algebra is projective by definition and since there always exists a homomorphism $f : \mathbf{A} \to \mathbf{T}$ for any $\mathbf{A} \in \mathsf{V}$, this means that any algebra in V is unifiable. So the Baker-Beynon property is equivalent to ask that every finitely presented algebra is weakly projective and it implies structurality; in the case of a variety being locally finite it also implies primitivity.

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A characterization of locally finite varieties of basic hoops was proved in 2000 by Blok and Ferreirim and it is not difficult to prove that

Theorem

Every locally finite variety of basic hoops has the Baker-Beynon property, hence it is primitive.

The PL_n^+ case

Another class of varieties that we managed to analyze is the one given by the varieties $\mathrm{PL}_n^+ := \mathbb{V}(\mathbf{Ł}_n^+ \oplus \mathbf{C}_\omega)$. Here our strategy involved finding a description of free algebras in this class of varieties, starting from the already known characterization of free algebras in $\mathrm{PL}_n := \mathbb{V}(\mathbf{Ł}_n \oplus \mathbf{C}_\omega)$, done by Busaniche and Cignoli.

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Theorem

For every finite set X it holds that

$$\mathbf{F}_{\mathrm{PL}_n^+}(X) \cong \Big(\prod_{\substack{S \in \mathrm{Ups}(X^{\gamma\sigma}) \\ S \neq X^{\gamma\sigma}}} \mathbf{Ł}_{k_S}^+ \oplus \mathbf{F}_\mathrm{C}(X_S) \Big) \times \mathbf{F}_\mathrm{C}(X),$$

for some k_S that divides n and some suitable $X_S \subseteq X$.

Structural completeness of PL_n^+

Have a representation of finitely generated free algebras, we can work with finitely presented algebras in PL_n^+. Using the fact that PL_n^+ is congruence distributive and that $\mathbb{V}(\mathbf{Ł}_n)$ and C have the Baker-Beynon property, it is not difficult to show that

Theorem

For every n, PL_n^+ has the Baker-Beynon property, hence it is structural.

Structural completeness in BL-algebras

Let us now analyze the case of BL-algebras. Here in general the Baker-Beynon property does not imply structurality, because not every BL-algebra is unifiable. However

Lemma

Let W *be a variety of basic hoops.*

- *If* W *has the Baker-Beynon property then* $\mathbb{V}(\text{Ł}_n) \oplus^t \mathsf{W}$ *has the Baker-Beynon property.*
- *Every nontrivial algebra in* $\mathbb{V}(\text{Ł}_1) \oplus^t \mathsf{W}$ *is unifiable.*

Hence, if W *has the Baker-Beynon property then* $\mathbb{V}(\text{Ł}_1) \oplus^t \mathsf{W}$ *is structural.*

Locally finite varieties

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Theorem

For a locally finite variety V *of* BL*-algebras the following are equivalent:*

1. V *is primitive;*
2. V *is structural;*
3. $\mathsf{V} = \mathbb{V}(\text{Ł}_1) \oplus^t \mathsf{W}$, *where* W *is a locally finite variety of basic hoops.*

Liftings of PL_n^+

In the non locally finite case, we can not say much. However, liftings the results about $\mathrm{PL_n}^+$ we get

Theorem

The varieties $\mathbb{V}(\mathbf{Ł}_1) \oplus^t \mathrm{PL_n}^+$ are structural for every n.

Quasivarieties

Quasivarieties are way more complex to deal with for several reasons, the main one is that a quasivariety Q in general is no longer generated by chains but rather by its Q-subdirectly irreducible algebras that do not need to be chains. For this reason we will restrict the majority of our analysis to quasivarieties generated by chains, where we will be able to start from some results about Wajsberg algebras and Wajsberg hoops.

Remember that, if Q is generated by a class of algebras K, by a result of Czelakowski and Dziobiak we have that every Q-subdurectly irreducible algebra is in $\mathbb{ISP}_u(\mathsf{K})$.

Wajsbeg algebras and Wajsberg hoops

Combining this fact with the characterization of structural quasivarieties in Wajsberg algebras (done by Gispert) and Wajsberg hoops we have the following corollary:

Corollary

In WA*:*

1. *if* $n \neq 1$*, then* $\mathbb{V}(\text{Ł}_n) = \mathbb{V}(\text{Ł}_1 \times \text{Ł}_n)$ *but* $\text{Ł}_n \notin \mathbb{Q}(\text{Ł}_1 \times \text{Ł}_n)$*, that is equivalent to say that* $\text{Ł}_n \notin \mathbb{ISP}_u(\text{Ł}_1 \times \text{Ł}_n)$*;*
2. *if* $n \neq 1$*, then* $\mathbb{V}(\text{Ł}_{n,k}) = \mathbb{V}(\text{Ł}_1 \times \text{Ł}_{n,1})$ *but* $\text{Ł}_{n,k} \notin \mathbb{Q}(\text{Ł}_1 \times \text{Ł}_{n,1})$*, that is equivalent to say that* $\text{Ł}_{n,k} \notin \mathbb{ISP}_u(\text{Ł}_1 \times \text{Ł}_{n,1})$*;*

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Quasivarieties of BL-algebras

Now, let us show how we can lift some of these results in order to say something about structural quasivarieties in BL and BH. For BL-algebras, the main idea is encoded in this rather technical lemma.

Lemma

Let $\mathbf{A} = \mathbf{B} \oplus \bigoplus_{i>0} \mathbf{B}_i$ *be a* BL*-chain; if* $\mathbf{A} \in \mathbb{ISP}_u(\mathbf{Ł}_1 \times \mathbf{A})$ *then* $\mathbf{B} \in \mathbb{ISP}_u(\mathbf{Ł}_1 \times \mathbf{B})$.

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This immediately gives us that

Theorem

Let $\mathbf{A}$ *be a* BL*-chain isomorphic to* $\mathbf{B} \oplus \bigoplus_{i>0} \mathbf{B}_i$*, where* $\mathbf{B} \cong \mathbf{Ł}_n$ *or* $\mathbf{B} \cong \mathbf{Ł}_{n,k}$ *with* $n \neq 1$*; then* $\mathbb{Q}(\mathbf{A})$ *is not structural.*

Quasivarieties

Moreover, starting from point 3 of the previous corollary, we can prove the following:

Theorem

Let $\mathbf{A}$ *be a* BL*-chain (or a* BH*-chain) isomorphic to* $\bigoplus_{i=0}^{n} \mathbf{B}_i \oplus \mathbf{B}$, *where* $\mathbf{B} \cong \mathbf{Ł}_{n,k}^{+}$ *with* $n, k \neq 1$*; then* $\mathbb{Q}(\mathbf{A})$ *is not structural;*

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The previous results show that, when considering if $\mathbb{Q}(\mathbf{A})$ is structural, the first (in the case of BL-algebras) and the last Wajsberg component of $\mathbf{A}$ play a special role. The idea is that for all other components of $\mathbf{A}$, we are tied up by how homomorphic images of ordinal sums work.

Further investigations and remarks

Although we have obtained some important results, the problem of structural completeness in BL and BH remains far from solved. There are two main reasons that make a complete characterization of all structural quasivarieties incredibly challenging.

- The first reason is that we lack a clear representation for free algebras in subvarieties of BL and BH. This is crucial because structural quasivarieties are precisely the ones generated by free algebras over countably many generators. The problem is that we still do not have a general method for representing $\mathbf{F}_{\mathbb{V}(\mathbf{A}\oplus\mathbf{B})}$ starting from $\mathbf{F}_{\mathbb{V}(\mathbf{A})}$ and $\mathbf{F}_{\mathbb{V}(\mathbf{B})}$. Understanding this will probably allow one to solve the problem of structural completeness in BL and BH building on the known solution about WA and WH.

  Another, perhaps simpler, problem concerning free algebras is that we would like to describe $\mathbf{F}_{\mathbb{V}(\mathbf{A},\mathbf{B})}$ starting from $\mathbf{F}_{\mathbb{V}(\mathbf{A})}$ and $\mathbf{F}_{\mathbb{V}(\mathbf{B})}$.

Further investigations and remarks

- The second major obstacle on our path were quasivarieties that are not generated by chains. It is known that the lattices of subquasivarieties of BL and BH are not numerable and that there exist quasivarieties which cannot be generated by chains. This is a problem because very little is known about them.

  For what concern structural completeness, what matters are structural cores, i.e. the smallest quasivariety generating a given variety, and when these structural cores are quasivarieties not generated by chains it is very difficult to describe them.

  One possible step could be trying to understand if there are portions of the lattices of quasivarieties where quasivarieties not generated by chains do not arise, like in the locally finite case.

Thanks for your attention!

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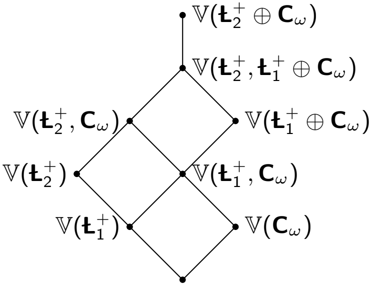

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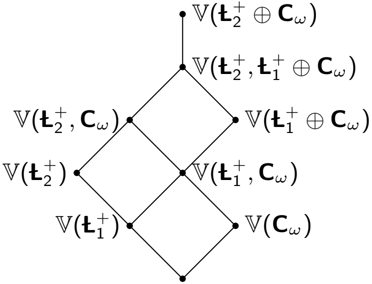

The problem is that we do not know if $\mathbb{V}(\mathbf{Ł}_2^+, \mathbf{Ł}_1^+ \oplus \mathbf{C}_\omega)$ is primitive or even structural. Note that $\mathbb{V}(\mathbf{Ł}_2^+)$ and $\mathbb{V}(\mathbf{Ł}_1^+ \oplus \mathbf{C}_\omega)$ are both structural, so in particular they are equal to $\mathbb{Q}(\mathbf{Ł}_2^+)$ and $\mathbb{Q}(\mathbf{Ł}_1^+ \oplus \mathbf{C}_\omega)$ respectively, and this implies that also $\mathbb{V}(\mathbf{Ł}_2^+, \mathbf{Ł}_1^+ \oplus \mathbf{C}_\omega) = \mathbb{Q}(\mathbf{Ł}_2^+, \mathbf{Ł}_1^+ \oplus \mathbf{C}_\omega)$.
Since $\mathbb{Q}(\mathbf{Ł}_2^+, \mathbf{Ł}_1^+ \oplus \mathbf{C}_\omega)$ is the smallest quasivariety generated by chains that generates $\mathbb{V}(\mathbf{Ł}_2^+, \mathbf{Ł}_1^+ \oplus \mathbf{C}_\omega)$, either $\mathbb{V}(\mathbf{Ł}_2^+, \mathbf{Ł}_1^+ \oplus \mathbf{C}_\omega)$ is structural or its structural core is not generated by chains.