Symmetric Difference in Lattices with Complementation

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Boolean lattices

In a Boolean lattice the symmetric difference is

$$x + y = (x \wedge y') \vee (x' \wedge y)$$

or

$$x + y = (x \vee y) \wedge (x' \vee y').$$

Three facts

- the two displayed expressions coincide;
- $+$ is associative;
- $x + y = 0$ exactly when $x = y$.

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The talk asks: how much of this Boolean behaviour survives beyond Boolean lattices, and in what form?

Our setting

A lattice with complementation is

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Throughout the slides, $\mathbf{L}$ denotes the algebra, while L denotes its lattice reduct or universe.

Coincidence identity

In every lattice with complementation, we can define

$$x +_1 y := (x' \wedge y) \vee (x \wedge y'), \qquad x +_2 y := (x \vee y) \wedge (x' \vee y').$$

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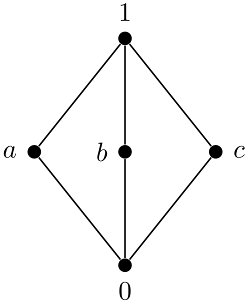
In Boolean lattices $+_1 = +_2$. Outside distributivity this is no longer automatic. Consider the coincidence identity:

$$(x' \wedge y) \vee (x \wedge y') \approx (x \vee y) \wedge (x' \vee y'). \tag{C}$$



x	0	a	b	c	1
x'	1	b	c	a	0

For the diamond $\mathbf{M}_3'$ with this complementation,

$$a +_1 b = b, \qquad a +_2 b = 1.$$

Being complemented and modular does not guarantee (C).

What (C) already forbids

Theorem

In every nontrivial lattice with complementation satisfying (C), no element can be a complement of both a and a'.

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Thus $\mathbf{M}_3'$ is excluded as a subalgebra:

Corollary

A lattice $(L, \vee, \wedge, ', 0, 1)$ with complementation satisfying identity (C) cannot contain a subalgebra isomorphic to $(M_3, \vee, \wedge, ', 0, 1)$.

Assuming modularity already gives Boolean:

Every modular lattice with complementation satisfying (C) is Boolean.

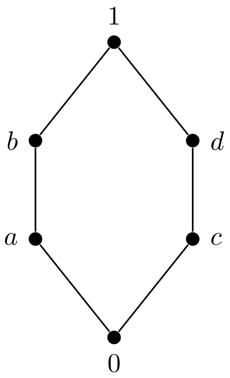
Coincidence: Horizontal sum of chains

Start with bounded chains C_i sharing only 0 and 1. Their horizontal sum glues the chains at these two endpoints:

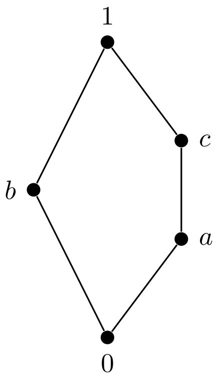
$$L = \bigcup_i C_i, \qquad \leq_L = \bigcup_i \leq_i \ .$$

If x, y lie in different chains and are not $0, 1$, then $x \wedge y = 0$ and $x \vee y = 1$.

Coincidence: Horizontal sum of chains



$$O_6 = C_4 \boxplus C_4$$



$$N_5 = C_3 \boxplus C_4.$$

Coincidence: Horizontal sum of chains

Theorem

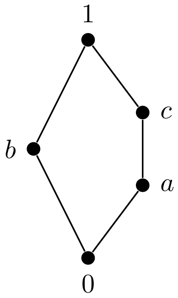
Let $L = C_1 \boxplus C_2$ and put $C_i^\circ = C_i \setminus \{0,1\}$. Then

$$' \text{ is a complementation} \iff \begin{cases} 0' = 1, \quad 1' = 0, \\ (C_1^\circ)' \subseteq C_2^\circ, \quad (C_2^\circ)' \subseteq C_1^\circ. \end{cases}$$

For every complementation of this form, the coincidence identity (C) holds. Moreover, De Morgan's laws hold exactly when $'$ is antitone.

Coincidence: Horizontal sum of chains

There are many non-modular examples satisfying (C) that arise this way and with different possible choices of complementations:



$N_5 = C_3 \boxplus C_4$

Antitone, but not involutive:

x	0	a	b	c	1
x'	1	b	a	b	0

,

another possible complementation:

x	0	a	b	c	1
x'	1	b	c	b	0

.

Both choices give subdirectly irreducible algebras.

Lexicographic sums of bounded lattices

Let B be a bounded lattice. For each $b \in B$, let F_b be a bounded lattice. The lexicographic sum of the family $(F_b)_{b \in B}$ is

$$L = \sum_{b \in B}^{\text{lex}} F_b = \{(b, x) \mid b \in B,\ x \in F_b\},$$

ordered by

$$(b, x) \leq (c, y) \iff b <_B c \text{ or } \big(b = c \text{ and } x \leq_{F_b} y\big).$$

We call F_b the fiber over b.

Lexicographic sums of bounded lattices

For $(b,x),(c,y) \in L$,

$$(b,x) \wedge (c,y) = \begin{cases} (b, x \wedge_{F_b} y), & b = c, \\ (b,x), & b <_B c, \\ (c,y), & c <_B b, \\ (b \wedge c, 1_{F_{b \wedge c}}), & b \parallel_B c. \end{cases}$$

The join is given dually.

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Slatinský (1994)

Assume that B is nontrivial. Then L is complemented if and only if B is complemented and the fibers F_{0_B} and F_{1_B} are one-element lattices.

Lexicographic sums with complementation

Let $\mathbf{B} = (B, \vee, \wedge, ', 0_B, 1_B)$ be a nontrivial lattice with complementation, and assume $F_{0_B} = \{\bullet_0\}$, $F_{1_B} = \{\bullet_1\}$. For each $b \in B \setminus \{0_B, 1_B\}$, choose any map $c_b : F_b \to F_{b'}$. Define

$$(b, x)' = (b', c_b(x)), \qquad (0_B, \bullet_0)' = (1_B, \bullet_1), \qquad (1_B, \bullet_1)' = (0_B, \bullet_0).$$

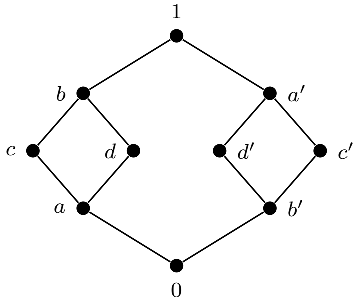
For an interior b, we have $b \parallel_B b'$; hence

$$(b, x) \wedge (b, x)' = (0_B, \bullet_0), \qquad (b, x) \vee (b, x)' = (1_B, \bullet_1).$$

Thus $'$ is a complementation of L.

Lexicographic sums with complementation: examples

The following examples are ortholattices satisfying (C) that can be described as lexicographic sums.



In these examples, the non-trivial fibers carry antitone involutions: after identifying F_b with $F_{b'}$, the map $x \mapsto x'$ becomes an antitone involution.

Open direction

Idea

(C) is one Boolean identity that survives in many non-Boolean lattices. Can this surviving Boolean behaviour be seen structurally?

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Lexicographic examples suggest a possible decomposition:

$$\text{Boolean skeleton} \quad + \quad \text{fibers over its elements.}$$

Questions

- Could lexicographic constructions of this kind describe a useful subclass of lattices with complementation satisfying (C), or perhaps even the whole class?
- How does the picture change if the complementation is additionally required to be antitone, involutive, or both?

Associativity of symmetric difference

Theorem

For a lattice with complementation $\mathbf{L}$,

$$\mathbf{L} \text{ is Boolean} \iff +_1 \text{ is associative} \iff +_2 \text{ is associative.}$$

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Why is associativity so restrictive?

Assume $+$ denotes either $+_1$ or $+_2$ and is associative. Then

$$x + 0 \approx x, \qquad x + 1 \approx x', \qquad x + x \approx 0.$$

Associativity gives $(x + y)' \approx x + y' \approx x' + y$ and $x'' \approx x$. The computation then reaches one of the Boolean characterization identities.

A two-variable Boolean test

Theorem

Let $\mathbf{L}$ be a lattice with complementation, and let $+ \in \{+_1, +_2\}$. Then

$$\mathbf{L} \text{ is Boolean} \iff (x+y)+y \approx x.$$

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$$\mathbf{L} \text{ is Boolean} \quad \Longleftrightarrow \quad (x+y)+y \approx x.$$

The key computation reduces the identity to a Boolean characterization theorem:

Theorem

Let $\mathbf{L}$ be a lattice with complementation. If $\mathbf{L}$ satisfies one of the identities

$$x \wedge y \approx x \wedge (x' \vee y), \qquad x \vee y \approx x \vee (x' \wedge y),$$

then $\mathbf{L}$ is distributive, and hence Boolean.

Equality detection

The symmetric difference can also be used to detect equality:

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The symmetric difference can also be used to detect equality:

$$x + y = 0 \quad \Longleftrightarrow \quad x = y.$$

For $+_1$, the non-trivial direction is the quasi-identity:

$$(x' \wedge y = 0 \ \& \ x \wedge y' = 0) \quad \Longrightarrow \quad x = y. \tag{W}$$

Let $\mathcal{W}$ denote the quasivariety of lattices with complementation satisfying (W).

Equality detection: first consequences

Once (W) holds, the chosen complementation cannot collapse elements.

Consequences of (W)

If $\mathbf{L} \in \mathcal{W}$, then:

- the chosen complementation is injective;
- more generally, if $x \leq y$ and $x' = y'$, then $x = y$;
- hence no member of $\mathcal{W}$ contains a subalgebra whose lattice reduct is N_5.

Equality detection

Boolean characterization

A lattice with complementation satisfies

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Length-two lattices

- $\mathbf{M}_2' \cong \mathbf{2}^2$ and $\mathbf{M}_3'$ belong to $\mathcal{W}$.
- For every cardinal $\kappa \geq 4$, no complementation on M_κ makes it a member of $\mathcal{W}$.

Equality detection with De Morgan laws

Weak orthomodularity follows

Every lattice with complementation satisfying (W) and De Morgan's laws is weakly and dually weakly orthomodular:

$$x \leq y \Rightarrow x \vee (x' \wedge y) \approx y, \qquad x \geq y \Rightarrow x \wedge (x' \vee y) \approx y.$$

Equality detection with De Morgan laws

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Modular lattices with complementation also satisfy both weak orthomodularity laws.

Equality detection: many non-Boolean examples

A semidirect-product-like construction

Let K be a complemented lattice and let $\mathbf{B}$ be a Boolean lattice. For each Boolean coordinate $x \in B$, choose a complementation S_x on K. Then

$$(a, x)' := (S_x(a), x')$$

defines a complementation on the product lattice $K \times B$.

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- The K-coordinate may use a different complementation over each $x \in B$.
- Conversely, every complementation on $K \times B$ is obtained this way.

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For $K = M_3$ and finite Boolean B, suitable choices give infinitely many finite subdirectly irreducible algebras in $\mathcal{W}$.

Further results

The quasivariety $\mathcal{W}$ and related classes are studied further in:

V. Cenker, I. Chajda, J. Kühr and H. Länger, *Varieties and quasivarieties of lattices with complementation*, arXiv:2605.17274, 2026.

Beyond this talk

The paper also studies axiomatizations, free algebras, and subdirectly irreducible members of related varieties and quasivarieties.
More on that at Jan Kühr talk tomorrow at 11.40, room 5.

Takeaway

- Beyond Boolean lattices, the usual properties of symmetric difference split apart.
- Constructions with a Boolean skeleton or Boolean factor, such as lexicographic sums and semidirect-like products, suggest that some non-Boolean examples may still be structurally controlled by Boolean data.

Thank you!

Questions?

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