# Layer decomposition for balanced residuated chains

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*TACL 2026, July 27, 2026*

# In the abstract

- We identify the broadest class of balanced residuated chains admitting a direct-system representation.

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# Layer decomposition for balanced residuated po-semigroups

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- We show that, in the finite totally ordered semigroup setting, the general theory collapses to a rigid direct-system representation, closely connected with Clifford-type ordinal sums.

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- We show that, in the finite totally ordered monoid setting, the general theory collapses to a rigid direct-system representation, closely connected with Clifford-type ordinal sums.

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- We show that, in the finite totally ordered monoid setting, the general theory collapses to a rigid direct-system representation, closely connected with Clifford-type ordinal sums.

# Historical background

- S. Jenei. Group representation for even and odd involutive commutative residuated chains. Studia Logica 110 (2022), 881–922. First circulated as arXiv:1910.01404 (2019). https://doi.org/10.1007/s11225-021-09981-y https://doi.org/10.48550/arXiv.1910.01404

- local unit function: $\boldsymbol{\tau} : X \to Id^{+}(X)$ $\boldsymbol{\tau}(x) = x \to x$

- Layer stratification: $L_u = \{ x \in X : \boldsymbol{\tau}(x) = u \}$

- Induced direct system: $(L_u, \rho_{u \to v})$, $\rho_{u \to v}(x) = xv$

- Reconstruction: the product by the Płonka-sum construction, the order by the directed-lexicographic order rule. The residual operation is determined by these.

# Historical background


- Jipsen, P., Tuyt, O., and Valota, D. The structure of finite commutative idempotent involutive residuated lattices. Algebra Universalis 82, 57 (2021).
- Jipsen, P., and Sugimoto, M. On varieties of residuated po-magmas and the structure of finite ipo-semilattices. In Topology, Algebra, and Categories in Logic 2022, Coimbra — Book of Abstracts, 2022.
- Gil-Férez, J., Jipsen, P., and Sugimoto, M. Locally integral involutive PO-semigroups. arXiv:2310.12926, 2023.
- Gil-Férez, J., Jipsen, P., and Lodhia, S. The structure of locally integral involutive po-monoids and semirings. In R. Glück, L. Santocanale, and M. Winter, editors, Relational and Algebraic Methods in Computer Science, RAMiCS 2023, Lecture Notes in Computer Science, vol. 13896, Springer, Cham, 2023.
- Bonzio, S., Gil-Férez, J., Jipsen, P., Přenosil, A., and Sugimoto, M. On the structure of balanced residuated partially ordered monoids. In Relational and Algebraic Methods in Computer Science, RAMiCS 2024, Lecture Notes in Computer Science, pp. 83–100, 2024.
- Sugimoto, M. Decompositions of locally integral involutive residuated structures. In Abstracts of the Topology, Algebra, and Categories in Logic 2024 Conference, Barcelona, 2024.
- Jipsen, P. On the structure of balanced residuated posets. In Abstracts of the Topology, Algebra, and Categories in Logic 2024 Conference, Barcelona, 2024.
- Bonzio, S., Gil-Férez, J., Jipsen, P., Přenosil, A., and Sugimoto, M. Balanced residuated partially ordered semigroups. arXiv:2505.12024, 2025.

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# Commutative setting

- $$\boldsymbol{\tau}(x) = x \to x$$

# Balanced setting

- $$\boldsymbol{\tau}(x) = x \to x$$
- $$x \backslash x = x / x$$

# Balanced setting

- $$\tau(x) = x \rightarrow x$$
- $$\tau(x) = x \backslash x = x / x$$

# The minimal conditions beyond balance to admit a direct system representation

- $\tau(xy) = \tau(x)\tau(y) \qquad \tau(xy) = \tau(x) \vee \tau(y)$

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- $$\begin{aligned} \tau(xy) &= \tau(x)\tau(y) & \tau(xy) &= \tau(x) \vee \tau(y) \\ \tau(x \backslash y) &= \tau(x)\tau(y) & \tau(x \backslash y) &= \tau(x) \vee \tau(y) \\ \tau(x / y) &= \tau(x)\tau(y) & \tau(x / y) &= \tau(x) \vee \tau(y) \end{aligned}$$

# The minimal conditions beyond balance are quite restrictive

| $n$ | TOCM | CRC | CRC$\tau(\cdot)$ | CRC$\tau(\rightarrow)$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 1 | 1 | 1 |
| 3 | 6 | 3 | 3 | 3 |
| 4 | 22 | 11 | 10 | 9 |
| 5 | 92 | 46 | 37 | 33 |
| 6 | 426 | 213 | 153 | 137 |
| 7 | 2146 | 1073 | 701 | 636 |
| 8 | 11634 | 5817 | 3534 | 3254 |
| 9 | 67472 | 33736 | 19491 | 18200 |
| 10 | 417654 | 208825 | 117255 | 110851 |

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| $n$ | TOCM | CRC | CRC$\tau(\cdot)$ | CRC$\tau(\to)$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 |
| 2 | 2 | 1 | 0 | 0 |
| 3 | 6 | 3 | 0 | 0 |
| 4 | 22 | 11 | 1 | 0 |
| 5 | 92 | 46 | 5 | 1 |
| 6 | 426 | 213 | 21 | 5 |
| 7 | 2146 | 1073 | 88 | 23 |
| 8 | 11634 | 5817 | 385 | 105 |
| 9 | 67472 | 33736 | 1788 | 497 |
| 10 | 417654 | 208825 | 8875 | 2471 |

# The minimal conditions beyond balance

## The topic of today's talk

- We identify the broadest class of balanced residuated po-semigroups admitting a direct-system representation.

to admit a direct system representation
are …

# … are these

# From primitive exactness to quotient-level coherence

# The canonical $\tau$-residuation-coherent partition

- Let $S=S_k(\mathbf{M})=\mathrm{Id}^+(\mathbf{M})$.
- **Definition** For a skeleton partition $\Theta$ and a block $A \in \Theta$, put $M_A := \{x \in M : \tau(x) \in A\}$.
  This is a union of primitive $\tau$-layers

# The canonical $\tau$-residuation-coherent partition

- **Definition** (Operation-output set) Let $\Theta$ be a partition of S, and let A,B $\in \Theta$. The operation-output set of the pair (A,B) is $\mathrm{Out}_\Theta(A,B) := \{\tau(xy), \tau(x\backslash y), \tau(x/y) : x \in M_A, y \in M_B\}$.
- It records, at the level of $\tau$-values, where multiplication, the left residual, and the right residual send pairs of elements whose input $\tau$-values lie in A and B.

# The canonical $\tau$-residuation-coherent partition

- **Definition** ($\tau$-multiplication and $\tau$-residuation coherence) A skeleton partition $\Theta$ is *$\tau$-residuation-coherent* if, for every pair of blocks $A,B\in\Theta$, the set
$$\mathrm{Out}_\Theta(A,B) = \{\tau(xy),\ \tau(x\backslash y),\ \tau(x/y) : x\in M_A,\ y\in M_B\}$$
is contained in a single $\Theta$-block.

- It is *$\tau$-multiplication-coherent* if, for every pair of blocks $A,B\in\Theta$, the set
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is contained in a single $\Theta$-block.

# The canonical $\tau$-residuation-coherent partition

- **Definition** (Residual connected-component coarsening) Let $\Theta$ be a skeleton partition of S. Define a graph on the set of blocks of $\Theta$ by joining two distinct blocks $C,D \in \Theta$ whenever there exist blocks $A,B \in \Theta$ such that $C,D \in \mathrm{Out}_{\Theta}(A,B)$.

- We denote by $CC_r(\Theta)$ the skeleton partition obtained by merging precisely those blocks that lie in the same connected component of this graph.

# The canonical $\tau$-residuation-coherent partition

- **Construction** (The canonical ω-limit τ-residuation closure) Let $\Theta_0$ be the singleton partition of S. Define recursively

$$\Theta_{n+1=}\ CC_r(\Theta_n) \qquad (n<\omega),$$

and let $\equiv_n$ be the equivalence relation on S whose classes are the blocks of $\Theta_n$. Since every step only merges blocks, the relations $\equiv_n$ form an increasing sequence. Set

$$e \equiv^r_\omega f \text{ iff } e \equiv^r_n f \text{ for some } n<\omega.$$

The union relation $\equiv^r_\omega$ is an equivalence relation (because an increasing union of equivalence relations is again an equivalence relation). Define

$$C_r(\mathbf{M}) := S/\equiv^r_\omega.$$

We call $C_r(\mathbf{M})$ the canonical *τ-residuation-coherent partition* of $S=S_k(\mathbf{M})$, or the *residual coherentization* of S.

# The canonical $\tau$-residuation-coherent partition

- **Proposition** The partition $C_r(\mathbf{M})$ is $\tau$-residuation-coherent. Moreover, $C_r(\mathbf{M})$ is the finest $\tau$-residuation-coherent skeleton partition of S. Equivalently, every $\tau$-residuation-coherent skeleton partition $\Phi$ satisfies
$$C_r(\mathbf{M}) \leq \Phi.$$
$C_r(\mathbf{M})$ is also $\tau$-multiplication-coherent.
The quotient $S/C_r(\mathbf{M})$ is a join-semilattice.

# Components, quotient skeleton, and shadows

# Component algebras

- Let $\Omega := C_r(\mathbf{M})$. For each block $A \in \Omega$, define
  $M_A := \{x \in M : \tau(x) \in A\}$. These stratify the algebra.
  The $\tau$-residuation coherence of $\Omega$ implies that $M_A$ is closed under multiplication and under both residuals. Thus the canonical residual components are again balanced residuated partially ordered semigroups with the inherited operations.

- $$\mathbf{M_A} := \langle M_A, \leq_A, \cdot_A, \backslash_A, /_A \rangle$$

# The quotient skeleton

- The quotient skeleton is the join-semilattice
$$\mathrm{Sk}_{\Omega}(\mathbf{M}) := \mathrm{Sk}(\mathbf{M})\,/\,\Omega.$$
The elements of $\Omega$ are the positive-idempotent blocks.

- The order is the join order:
$$A \leq_{\Omega} B \quad \Leftrightarrow \quad A \vee B = B.$$

# Product and residual shadows

- **Definition** (Product and residual shadows) Let $A, C \in Sk_\Omega(\mathbf{M})$ with $A \leq_\Omega C$, and let $q \in C$. For $x \in M_A$, define

- $$\lambda^{A,C}{}_q(x) := xq, \qquad \eta^{A,C}{}_q(x) := x/q.$$

- When the source and target are clear, write simply $\lambda_q(x) = xq$, $\eta_q(x) = x/q$. The family $\{\lambda_q : q \in C\}$ is the *product-shadow family* from A into C, and the family $\{\eta_q : q \in C\}$ is the *residual-shadow family* from A into C.

- **Lemma** (Shadows land in the target component) If $x \in M_A$, $A \leq_\Omega C$, and $q \in C$, then $xq \in M_C$ , $x/q \in M_C$.

# Reconstruction

# Canonical resolved two-shadow system

The reconstruction data are

$$\mathfrak{D}_{\mathrm{r}}(\mathbf{M}) := \left(\mathrm{Sk}_{\mathrm{r}}(\mathbf{M}), \{L_A : A \in \mathrm{Sk}_{\mathrm{r}}(\mathbf{M})\}, \{\lambda_q^{A,C} : A \leq_\Omega C,\ q \in C\}, \{\eta_q^{A,C} : A \leq_\Omega C,\ q \in C\}\right),$$

where

$$\mathrm{Sk}_{\mathrm{r}}(\mathbf{M}) = \mathrm{Sk}_{\mathcal{C}_{\mathrm{r}}(\mathbf{M})}(\mathbf{M}),$$

and $\leq_\Omega$ denotes the order of this residual quotient skeleton, and, for $x \in L_A$,

$$\lambda_q^{A,C}(x) = xq, \qquad \eta_q^{A,C}(x) = x/q.$$

# Product recovery

- Let $x \in M_A$ and $y \in M_B$ , and put $C= A \vee B$. The resolved product candidates are

$$\Pi_C\,(x,y) := \{\lambda^{A,C}{}_q(x)\ \lambda^{B\,,C}{}_q(y) : q \in C\} \subseteq M_C.$$

- **Proposition** (Product recovery) The product $xy$ is the least element of $\Pi_C\,(x,y)$:

$$xy = \min \Pi_C\,(x,y).$$

# Order recovery

- **Theorem** (Resolved order-completeness) Every $\tau$-residuation-coherent skeleton partition satisfies the resolved order formula: for all $x \in M_A$ and $y \in M_B$ , with $C=A\vee B$,

$$x\leq y \quad \Leftrightarrow \quad \exists q\in C: \quad xq \leq_C y/q$$

$$x\leq y \quad \Leftrightarrow \quad \exists q\in C: \quad \lambda^{A,C}{}_q(x) \leq_C \eta^{A,C}{}_q(x).$$

- **Corollary** (Order recovery for the residual coherentization) For $\Omega = C_r(\mathbf{M})$ the above holds.

# Collapse to ordinary direct systems

- If every block has a least element and these satisfy a natural monotonicity condition, then the shadow data are no longer needed.
- The two-shadow system reduces to an ordinary direct-system presentation.
- More precisely: A two-shadow system collapses to a two-transition direct system.

# The recovery theorem

**Theorem** (Resolved $\tau$-residuation-coherent decomposition–reconstruction) *Let* $\mathbf{M}$ *be a balanced residuated ordered semigroup, and let* $\Omega = \mathcal{C}_{\mathrm{r}}(\mathbf{M})$. *Then* $\mathbf{M}$ *is recovered, up to the evident isomorphism, from the following data:*

(1) *the quotient skeleton* $\mathrm{Sk}_{\mathrm{r}}(\mathbf{M})$*;*

(2) *the component ordered semigroups* $\mathbf{M}_A$, *for* $A \in \mathrm{Sk}_{\mathrm{r}}(\mathbf{M})$*;*

(3) *the product shadows* $\lambda_q(x) = xq$, *where* $q$ *ranges over the target block;*

(4) *the residual shadows* $\eta_q(x) = x/q$, *where* $q$ *ranges over the target block.*

*The product is reconstructed as a least product-shadow value. The order is reconstructed by the resolved two-shadow formula. The residuals are then reconstructed, on their own, as the residuals determined by this recovered ordered semigroup.*

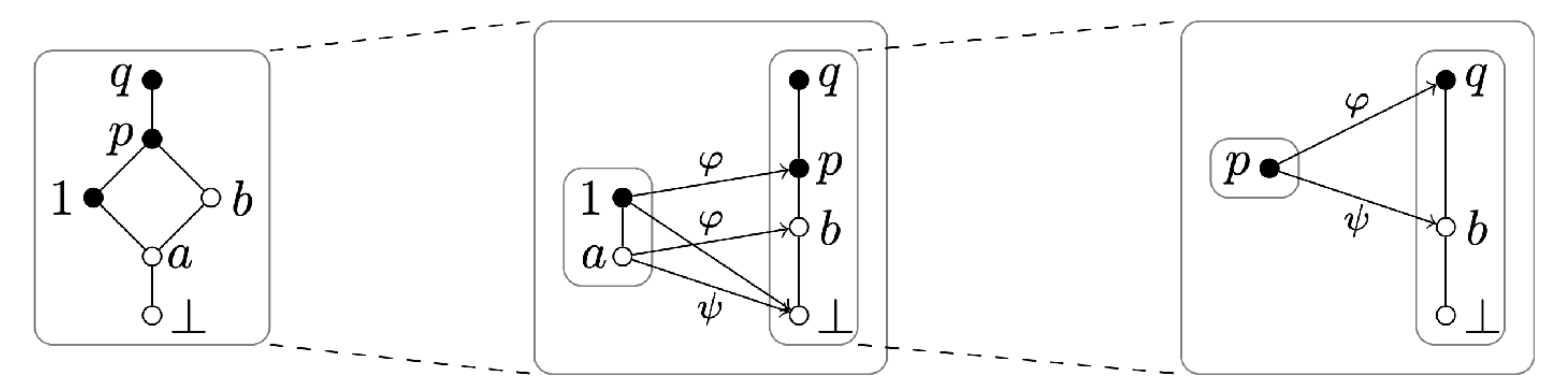
# Comparison with the BGJPS Płonka-sum method

# Agreement under strong exactness conditions

- $$\begin{aligned} \tau(xy) &= \tau(x)\tau(y) & \qquad \tau(xy) &= \tau(x) \vee \tau(y) \\ \tau(x\backslash y) &= \tau(x)\tau(y) & \qquad \tau(x\backslash y) &= \tau(x) \vee \tau(y) \\ \tau(x/y) &= \tau(x)\tau(y) & \qquad \tau(x/y) &= \tau(x) \vee \tau(y) \end{aligned}$$

- Under these assumptions, residual coherentization collapses to the BGJPS method: the canonical $\tau$-residuation-coherent partition of $\mathrm{Id}^{+}(\mathbf{M})$ becomes the partition into singletons, the two shadow systems reduce to two ordinary transition systems of direct systems, and the general recovery formulas specialize to the BGJPS formulas.

# Subsemilattice-steady visibility



# Residual coherentization refines subsemilattice-steady visibility - the first level

- **Theorem** (First-level comparison with subsemilattice-steady decompositions) Let **A** be a balanced residuated partially ordered semigroup, and let
$$I \subseteq \mathrm{Sk}(\mathbf{A})$$
be a nonempty subsemilattice over which **A** is steady in the sense of [BGJPS]. Then
$$C_r(\mathbf{A}) \leq P_I ,$$
where ≤ denotes refinement of partitions.
Equivalently, every component of the first-level $C_r$-decomposition is contained in a component of the subsemilattice-steady decomposition over I.
In particular, if Sk(**A**) is finite, then
$|C_r(\mathbf{A})| \geq |P_I| = |I|$.

# Residual coherentization refines subsemilattice-steady visibility - the frontier

- **Definition** (Frontier partitions of an iterated decomposition) Let a decomposition procedure be iterated on a balanced residuated partially ordered semigroup **A**. A component is *terminal* for the procedure if one more application of the procedure produces no proper refinement of that component. The frontier partitions F0,F1,F2,… are defined as follows. Put
  $F_0 = \{A\}$.
  Given $F_n$, each nonterminal component is replaced by the components produced by one application of the decomposition procedure inside it, while each terminal component is carried forward unchanged. The resulting partition of the original carrier is $F_{n+1}$. Thus $F_n$ is not the set of vertices at graph-theoretic depth n. It is the current frontier after n rounds of refinement. In particular, terminal leaves remain present in all later frontier partitions.

# Residual coherentization refines subsemilattice-steady visibility - the frontier

- **Theorem** (Frontier comparison for the residual tree) Let **A** be a balanced residuated po-semigroup. Let $(R_n)_{n\geq 0}$ and $(S_n)_{n\geq 0}$ be the frontier partitions of two iterated decompositions of the original carrier. Assume that $(R_n)_{n\geq 0}$ is obtained by iterating the residual coherentization $C_r$. Assume that $(S_n)_{n\geq 0}$ is obtained by iterating subsemilattice-steady decompositions in the sense of [BGJPS]: at each nonterminal step, a steady subsemilattice of the positive-idempotent skeleton of the current component is chosen, it is required to be steady over that subsemilattice, and the current component is replaced by the corresponding steady fibres. Then

$$R_n \leq S_n \ (n\geq 0).$$

Consequently, whenever the frontier partitions are finite, $|R_n| \geq |S_n|$ $(n\geq 0)$.
If the subsemilattice-steady iteration terminates with terminal frontier $S_N$, and the residual iteration also terminates with terminal frontier $R_M$, then $R_M \leq S_N$.
Equivalently, every terminal residual-coherent component is contained in a unique terminal subsemilattice-steady component, and every terminal subsemilattice-steady component is the disjoint union of the terminal residual-coherent components contained in it. In particular, each such terminal residual-coherent component is an induced balanced residuated component subalgebra of the terminal subsemilattice-steady component which contains it.
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  Consequently, whenever the frontier partitions are finite, $|R_M| \geq |S_N|$.

# Examples separating residual coherentization from the BGJPS (subsemilattice-steady) method

- Let **A** be the direct product **B** × **D**, where **B** is the complex residuated lattice of the two-element group, and **D** is the complex residuated lattice of the three-element residuated chain whose product is maximum.

  Then, at the frontier level,
  -$C_r$(**A**) has four blocks, whereas the subsemilattice-steady method yields only two,
  - $C_r$(**D**) has two blocks, whereas **D** is indecomposable for the subsemilattice-steady method.

# Recapitulation

- Instead of imposing restrictive identities that lead to a narrow scope, we work with the whole class of balanced residuated po-semigroups.
- We form the canonical $\tau$-residuation-coherent partition (the residual coherentization).
- From this partition we obtain the component semigroups, the quotient skeleton, and the product and residual shadows; together these form the resolved two-shadow system.
- The original algebra is recovered by the product-recovery rule, the resolved order-recovery rule, and the determinacy of the residuals from the recovered product and order.
- For every balanced residuated po-semigroup, the canonical residual coherentization yields a decomposition refining every subsemilattice-steady BGJPS decomposition; on suitable subclasses, the refinement is strict.

# The balanced finite totally ordered monoid setting - Clifford ordinal sum-like structure

- Scope: monoidal
- $C_m(\mathbf{M})$ - multiplicative coherentization
- component ordered semigroups
- **rigid** (single-transition) direct system

# The balanced finite totally ordered monoid setting - Clifford ordinal sum-like structure

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 29 |
| 3 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 29 |
| 4 | 1 | 2 | 2 | 2 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 29 |
| 5 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 28 | 29 |
| 6 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 27 | 28 | 29 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 6 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 8 | 8 | 27 | 28 | 29 |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 6 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 27 | 28 | 29 |
| 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 10 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 17 | 17 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 11 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 11 | 11 | 11 | 11 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 12 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 11 | 11 | 12 | 13 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 13 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 11 | 11 | 13 | 13 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 15 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 13 | 13 | 15 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 16 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 16 | 16 | 16 | 16 | 16 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 17 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 17 | 17 | 17 | 17 | 17 | 17 | 17 | 19 | 19 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 18 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 18 | 18 | 18 | 18 | 18 | 18 | 18 | 19 | 19 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 19 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 20 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 21 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 22 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 23 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 23 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 24 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 25 | 26 | 27 | 28 | 29 |
| 25 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 8 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 26 | 27 | 28 | 29 |
| 26 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 8 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 27 | 28 | 29 |
| 27 | 1 | 2 | 3 | 4 | 5 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 28 | 29 |
| 28 | 1 | 2 | 4 | 4 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 29 |
| 29 | 1 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 |

# The balan d monoid setting - C structure

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 29 |
| 3 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 29 |
| 4 | 1 | 2 | 2 | 2 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 29 |
| 5 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 28 | 29 |
| 6 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 27 | 28 | 29 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 6 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 8 | 8 | 27 | 28 | 29 |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 6 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 27 | 28 | 29 |
| 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 10 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 17 | 17 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 11 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 11 | 11 | 11 | 11 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 12 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 11 | 11 | 12 | 13 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 13 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 11 | 11 | 13 | 13 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 15 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 13 | 13 | 15 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 16 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 16 | 16 | 16 | 16 | 16 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 17 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 17 | 17 | 17 | 17 | 17 | 17 | 17 | 19 | 19 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 18 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 18 | 18 | 18 | 18 | 18 | 18 | 18 | 19 | 19 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 19 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 20 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 21 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 22 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 23 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 23 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 24 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 25 | 26 | 27 | 28 | 29 |
| 25 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 8 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 26 | 27 | 28 | 29 |
| 26 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 8 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 27 | 28 | 29 |
| 27 | 1 | 2 | 3 | 4 | 5 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 28 | 29 |
| 28 | 1 | 2 | 4 | 4 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 29 |
| 29 | 1 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 |

The b … onoid

settin … cture

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 29 |
| 3 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 29 |
| 4 | 1 | 2 | 2 | 2 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 29 |
| 5 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 28 | 29 |
| 6 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 27 | 28 | 29 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 6 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 7 | 8 | 8 | 27 | 28 | 29 |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 6 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 27 | 28 | 29 |
| 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 10 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 17 | 17 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 11 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 11 | 11 | 11 | 11 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 12 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 11 | 11 | 12 | 13 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 13 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 11 | 11 | 13 | 13 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 14 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 15 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 13 | 13 | 15 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 16 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 16 | 16 | 16 | 16 | 16 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 17 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 17 | 17 | 17 | 17 | 17 | 17 | 17 | 19 | 19 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 18 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 18 | 18 | 18 | 18 | 18 | 18 | 18 | 19 | 19 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 19 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 20 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 21 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 22 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 23 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 23 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 24 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 25 | 26 | 27 | 28 | 29 |
| 25 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 8 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 25 | 26 | 27 | 28 | 29 |
| 26 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 8 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 27 | 28 | 29 |
| 27 | 1 | 2 | 3 | 4 | 5 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 27 | 28 | 29 |
| 28 | 1 | 2 | 4 | 4 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 29 |
| 29 | 1 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 29 |

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| 29 | 28 | 27 | 28 | 27 | 26 | 24 | 26 | 19 | 16 | 15 | 14 | 15 | 14 | 15 | 16 | 16 | 16 | 19 | 20 | 21 | 21 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |

$$\begin{bmatrix} 1&1&1&1&1\\ 1&1&1&2&3\\ 1&1&1&3&3\\ 1&2&3&4&5\\ 1&3&3&5&5 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&1&1\\ 1&2&2&4&4&6\\ 1&2&3&4&5&6\\ 1&4&4&6&6&6\\ 1&5&5&6&6&6\\ 1&6&6&6&6&6 \end{bmatrix} \quad \begin{bmatrix} 1 \end{bmatrix} \quad \begin{bmatrix} 1&2&3\\ 2&3&3\\ 3&3&3 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&1&1\\ 1&1&1&2&3&3\\ 1&1&1&3&3&3\\ 1&2&3&4&5&6\\ 1&3&3&5&5&6\\ 1&3&3&6&6&6 \end{bmatrix} \quad \begin{bmatrix} 1&1&1&1&1&1\\ 1&1&1&2&2&3\\ 1&1&1&3&3&3\\ 1&2&3&4&4&6\\ 1&2&3&4&5&6\\ 1&3&3&6&6&6 \end{bmatrix} \quad \begin{bmatrix} 1&1\\ 1&2 \end{bmatrix}$$

# Coming to arXiv in a Few Days

- SJ, A canonical rigid direct-system representation of finite local-unit-aligned totally ordered monoids
- SJ, Canonical resolved-transport decomposition and reconstruction of local-unit-aligned ordered semigroups
- SJ, Residual coherentization of balanced residuated partially ordered semigroups

That is all