Algebraic Semantics for First-Order Relevant Logics

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Fundação para a Ciência e a Tecnologia

language, mind and cognition

Outline

Initial Remarks

What Semantics There Are (and Aren't)

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- Other semantics have been developed.

Currently no 'hard' algebraic semantics (except Ferenz and Lin's forthcoming paper on Polyadic De Morgan Monoids).

Polyadic Relevant Algebras

Relevant Algebras

Definition (Relevant Algebras)

A *relevant algebra* is a tuple $\mathbf{A} = \langle A, \wedge, \vee, \circ, \rightarrow, \neg, 1 \rangle$ where

(r1) $\langle A, \wedge, \vee \rangle$ is a distributive lattice;

(r2) $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$;

(r3) $(b \vee c) \circ a = (b \circ a) \vee (c \circ a)$;

(r4) $\neg(a \vee b) = \neg a \wedge \neg b$;

(r5) $\neg(a \wedge b) = \neg a \vee \neg b$

(r6) $1 \circ a = a$;

(r7) $a \circ b \leq c$ iff $a \leq b \rightarrow c$;

where $\leq$ is the lattice ordering defined as usual.

Polyadic Relevant Algebras I

Definition (Polyadic Relevant Algebras)

Let I be a non-empty set. By a *polyadic (I)-relevant algebra* we mean an algebra of the form:

$$\mathbf{A} := \langle A, \wedge, \vee, \circ, \to, \neg, 1, \langle \forall_J, \exists_J \mid J \subseteq_\omega I \rangle, \langle S_\sigma \mid \sigma \in I^{(I)} \rangle \rangle$$

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satisfying the following equations (where $J, K \subseteq_\omega I$, $Q_J \in \{\forall_J, \exists_J\}$ and $\sigma, \tau \subseteq I^{(I)}$):[1]

(PA$_0$) $S_\sigma 1 = 1$;

(PA$_1$) $S_\delta x = x$;

(PA$_2$) $S_\sigma(S_\tau x) = S_{\sigma\tau} x$;

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(PA$_5$) $Q_J S_\sigma x = S_\sigma Q_{\sigma^{-1} J} x$, where σ is one-to-one on $\sigma^{-1} J$;

(RA) The equalities (r1)–(r7) defining relevant algebras; and (NEXT SLIDE)

Polyadic Relevant Algebras II: More Equalities

(Q0) $x \leq y$ implies $S_\sigma x \leq S_\sigma y$;
(Q1) $\forall_J 1 = 1 = \exists_J 1$;
(Q2) $\forall_\emptyset x = x = \exists_\emptyset x$;
(Q3) $\forall_{J \cup K} x = \forall_J \forall_K x$;
(Q4) $\exists_{J \cup K} x = \exists_J \exists_K x$;
(Q5) $\forall_J x \leq x$;
(Q6) $x \leq \exists_J x$;
(Q7) $\forall_J (x \wedge y) = \forall_J x \wedge \forall_J y$;
(Q8) $\exists_J (x \vee y) = \exists_J x \vee \exists_J y$;
(Q9) $\forall_J \forall_J x = \forall_J x = \exists_J \forall_J x$;
(Q10) $\exists_J \exists_J x = \exists_J x = \forall_J \exists_J x$;
(Q11) $\forall_J (\forall_J x \to \forall_J y) = \forall_J x \to \forall_J y = \exists_J (\forall_J x \to \forall_J y)$
(Q12) $\forall_J (\forall_J x \circ \forall_J y) = \forall_J x \circ \forall_J y = \exists_J (\forall_J x \circ \forall_J y)$
(Q13) $\forall_J (\forall_J x \leftarrow \forall_J y) = \forall_J x \leftarrow \forall_J y = \exists_J (\forall_J x \leftarrow \forall_J y)$
(Q14) $\forall_J \neg \forall_J x = \neg \forall_J x = \exists_J \neg \forall_J x$
(Q15) $x \leq y$ implies $x \leq \forall_J y$, where $x = \forall_J x = \exists_J x$;
(Q16) $x \leq y$ implies $\exists x \leq y$, where $y = \forall_J y = \exists_J y$;

Polyadic Relevant Algebras III: Some Special Algebras

As polyadic relevant algebra is said to be *confined* when it satisfies the following equalities (EC1) and (EC2).

(EC1) $\forall_J(x \vee y) = x \vee \forall_J y$; (where $x = \forall_J x = \exists_J x$)

(EC2) $\exists_J(x \wedge y) = x \wedge \exists_J y$; (where $x = \forall_J x = \exists_J x$)

- (We are interested in relevant logics as a set of theorems)

Theorem

For any first-order formula φ, and suitable mappings v:

$$1 \leq v(\varphi) \text{ in every PRA iff } \vdash_{\mathbf{QBM}} \varphi$$

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Proof.

Sketch: (1) The Lindembaum-Tarski algebra (using the congruence from provable bi-conditionals) for **QBM** is a polyadic relevant algebra. (2) Every proof in **QBM** of φ can be turned into a proof of $1 \leq v(\varphi)$. □

First Representation Theorem

RA of truth values

Definition

We define a *Relevant algebra of truth values* to be an algebra

$$\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \circ^{\mathbf{A}}, \rightarrow^{\mathbf{A}}, \leftarrow^{\mathbf{A}} \neg^{\mathbf{A}}, 1^{\mathbf{A}}, \forall^{\mathbf{A}}, \exists^{\mathbf{A}} \rangle$$

where $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \circ^{\mathbf{A}}, \rightarrow^{\mathbf{A}}, \leftarrow^{\mathbf{A}} \neg^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ is a relevant algebra, and $\forall^{\mathbf{A}}, \exists^{\mathbf{A}} : \wp(A) \rightharpoonup A$ is a partial operation from $\wp(A)$ to A.

Functional Polyadic Relevant Algebras

Definition (Functional Polyadic Relevant Algebras)

Given two sets D, I. A *partial functional polyadic relevant algebra* (FPRA) $\bar{\mathbf{A}}$ is of the form

$$\langle A^{D^I}, \wedge^{\bar{\mathbf{A}}}, \vee^{\bar{\mathbf{A}}}, \neg^{\bar{\mathbf{A}}}, \circ^{\bar{\mathbf{A}}}, \rightarrow^{\bar{\mathbf{A}}}, 1^{\bar{\mathbf{A}}}, \forall^{\bar{\mathbf{A}}}, \exists^{\bar{\mathbf{A}}}, S^{\bar{\mathbf{A}}}\rangle$$

where the type/arity of each arguments and the operators including $Q^{\bar{\mathbf{A}}}, : \mathcal{P}_\omega(I) \rightarrow [A^{D^I}, A^{D^I}]$ for $Q \in \{\forall, \exists\}$ and $S^{\bar{\mathbf{A}}} : I^I \rightarrow End(\mathbf{A})$ are defined as follows:

(f1) Some stuff for the non-quantifiers.

(f2) $(Q_J^{\bar{\mathbf{A}}} p)(\vec{x}) = Q^{\mathbf{A}}(\{p(\vec{y}) : \vec{x}J_*\vec{y}\})$, for all $Q \in \{\forall, \exists\}, p \in A^{D^I}, J \subseteq_\omega I$, and $\vec{x}, \vec{y} \in D^I$;

(f3) $(S_\sigma^{\bar{\mathbf{A}}} p)(\vec{x}) = p(\vec{x} \cdot \sigma)$ where $(\vec{x} \cdot \sigma)$ is the composition of $\vec{x}$ with σ (and therefore of type D^I).

with the following further required conditions:

(C1) $\forall^{\mathbf{A}}(\{p(\vec{y}) : \vec{x}J_*\vec{y}\}) = \bigwedge(\{p(\vec{y}) : \vec{x}J_*\vec{y}\})$; $\exists^{\mathbf{A}}(\{p(\vec{y}) : \vec{x}J_*\vec{y}\}) = \bigvee(\{p(\vec{y}) : \vec{x}J_*\vec{y}\})$

Functional Representation Theorem

Theorem

Every locally finite polyadic relevant algebra (of infinite dimension I) is isomorphic to a (function-representable) polyadic relevant algebra.

1. Pigozzi and Salibra's functional representation theorem for abstract variable binding calculi [3]
2. Used by Pigozzi and Salibra to give functional representation theorem for polyadic non-classical logics based on Rasiowa's *standard implicational calculi* [2]

Second Representation Theorem

Representation by Set-based Relational Frames

- PRAs are (multi-)Gaggles
- Gaggles are nicely represented by relational frames.

Representation by Set-based Relational Frames

- PRAs are (multi-)Gaggles
- Gaggles are nicely represented by relational frames.
- Generalizing Jonson and Tarski's BAOs
- Goldblatt's 'Varieties of Complex Algebras'
- In general: the relation between certain algebras and topological structures

Notes on Notation

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- $R_{S_\tau} \beta\alpha$: α contains at least all the $S_\tau a$ for each $a \in \beta$

and

- $R_{S_\tau} \hat{\beta}\alpha$ means $R_{S_\tau} \beta\alpha$ and for all γ, $R_{S_\tau} \gamma\alpha$ implies $\gamma \leq \beta$.

PRA frames I

Definition (PRA-Frames)

A *frame for polyadic relevant I-algebras* (or simply an I-Frame) is a tuple

$$\mathfrak{F} = \langle W, N, R_{\circ}, *, \{R_{S(\tau)}\}_{\tau \in I^{(I)}}, \{R_J\}_{J \subseteq_{\omega} I} \rangle$$

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- $\alpha \leq \beta$ means $\exists \gamma \in N(R_\circ \gamma \alpha \beta)$
- $\alpha \leq \beta$ implies $\beta^* \leq \alpha^*$
- N is an $\leq$-upset, and $R_\circ \downarrow\downarrow\uparrow$.

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- $\alpha \leq \beta$ implies $\beta^* \leq \alpha^*$
- N is an $\leq$-upset, and $R_\circ \downarrow\downarrow\uparrow$.
- $R_{S_\tau} \beta \alpha$ and $\alpha \leq \gamma$ imply $R_{S_\tau} \beta \gamma$ (S-Tonicity)
- $R_{S_\tau} \beta \alpha$ then $\beta \in N$ iff $\alpha \in N$ (S-Logicality)
- $R_{S_\delta} \beta \alpha$ iff $\beta \leq \alpha$ (S-Identity)
- If $R_{S_\tau} \beta \alpha$ then $\exists \gamma$ such that $\beta \leq \gamma$, $R_{S_\tau} \hat{\gamma} \alpha$, $R_{S_\tau} \hat{\gamma}^* \alpha^*$ (S-$\neg$-Dist)[2]

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- N is an $\leq$-upset, and $R_\circ$ $\downarrow\downarrow\uparrow$.
- $R_{S_\tau}\beta\alpha$ and $\alpha \leq \gamma$ imply $R_{S_\tau}\beta\gamma$ (S-Tonicity)
- $R_{S_\tau}\beta\alpha$ then $\beta \in N$ iff $\alpha \in N$ (S-Logicality)
- $R_{S_\delta}\beta\alpha$ iff $\beta \leq \alpha$ (S-Identity)
- If $R_{S_\tau}\beta\alpha$ then $\exists\gamma$ such that $\beta \leq \gamma$, $R_{S_\tau}\hat{\gamma}\alpha$, $R_{S_\tau}\hat{\gamma}^*\alpha^*$ (S-¬-Dist)[2]
- for each τ and $\alpha \in W$, we have $\exists\beta(R_{S_\tau}\beta\alpha)$ (S-Seriality)
- If $\exists\beta(R_{S_\tau}\beta\alpha)$ then $\exists\gamma R_{S_\tau}\hat{\gamma}\alpha$ (S-Maximality)

PRA Frames II: More Conditions

- If $R_{S_\tau \alpha' \alpha}$ then $R_\circ \alpha' \beta' \gamma'$ iff $\exists \beta\gamma(R_\circ \alpha\beta\gamma$ and $R_{S_\tau} \beta' \beta$ and $R_{S_\tau} \gamma' \gamma)$ (R-S-Invariance)
- $\exists \alpha' (R_{S_\tau} \alpha' \alpha$ and $R_\circ \beta' \gamma' \alpha')$ if and only if
  $\exists \beta\gamma(R_\circ \beta\gamma\alpha$ and $R_{S_\tau \beta' \beta}$ and $R_{S_\tau \gamma' \gamma})$ (R-S-Invariance 2)

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- $\exists\alpha'(R_{S_{\tau}}\alpha'\alpha$ and $R_{\circ}\beta'\gamma'\alpha')$ if and only if
  $\exists\beta\gamma(R_{\circ}\beta\gamma\alpha$ and $R_{S_{\tau}\beta'\beta}$ and $R_{S_{\tau}\gamma'\gamma})$ (R-S-Invariance 2)
- $R_{S_{\sigma}}\hat{\beta}\alpha$, $R_{S_{\tau}}\hat{\gamma}\alpha$, and $R_{J}\gamma\eta$ imply $R_{J}\beta\eta$, when $\sigma J_{*}\tau$ (CPA$_4\forall$)
- $R_{S_{\sigma}}\hat{\beta}\alpha$, $R_{S_{\tau}}\hat{\gamma}\alpha$, and $R_{J}\eta\beta$ imply $R_{J}\eta\gamma$, when $\sigma J_{*}\tau$ (CPA$_4\exists$)

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- $R_{S_\sigma}\hat{\beta}\alpha$, $R_{S_\tau}\hat{\gamma}\alpha$, and $R_J\eta\beta$ imply $R_J\eta\gamma$, when $\sigma J_* \tau$ (CPA$_4\exists$)
- $\exists \beta(R_J\beta\alpha$ and $R_{S_\sigma}\gamma\beta)$ if and only if $\exists\beta(R_{S_\sigma}\beta\alpha$ and $R_{(\sigma^{-1}J)}\gamma\beta$,
  when σ is one-to-one on $\sigma^{-1}J$ (CPA$_5\exists$)[3]
- When σ is one-to-one on $\sigma^{-1}J$:
  (1) $\exists\beta(R_{S_\sigma}\hat{\beta}\alpha$ and $R_{(\sigma^{-1}J)}\beta\gamma$ only if $\exists\beta(R_J\alpha\beta$ and $R_{S_\sigma}\gamma\beta)$, and
  (2) $\exists\beta(R_J\alpha\beta$ and $R_{S_\sigma}\hat{\gamma}\beta)$only if $\exists\beta(R_{S_\sigma}\hat{\beta}\alpha$ and $R_{(\sigma^{-1}J)}\beta\gamma$ (CPA$_5\forall$)

PRA Frames III: Even More Conditions

- $R_J bc$ and $a \leq b$ and $c \leq d$ imply $R_J ad$ — i.e. $R_J \downarrow\uparrow$ (Tonicity)
- $R_J \beta\alpha$ and $\beta \in N$ imply $\alpha \in N$ (Logicality)[4]

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- $R_\varnothing ab$ iff $a \leq b$ (Vacuity)

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- $R_J ab$ and $R_J bc$ imply $R_J ac$ (Transitivity)
- $R_J \alpha\alpha$ (Reflexivity)

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- $R_{J \cup K} ab$ iff $\exists c(R_K ac$ and $R_J cb)$ (Union Transitivity)
- $R_J ab$ and $R_J bc$ imply $R_J ac$ (Transitivity)
- $R_J \alpha\alpha$ (Reflexivity)
- $\exists\gamma(R\alpha\beta\gamma$ and $R_J\gamma\epsilon)$ implies $\exists\alpha', \beta'(R\alpha'\beta'\epsilon$ and $R_J\alpha\alpha'$ and $R_J\beta\beta')$[5] (K∀)
- $\exists\gamma(R\alpha\beta\gamma$ and $R_J\epsilon\beta)$ implies $\exists\alpha', \gamma'(R\alpha'\epsilon\gamma'$ and $R_J\alpha\alpha'$ and $R_J\gamma'\gamma)$ (K∃)

PRA models I

Definition (PRA-Models)

A *polyadic relevant algebra model* for polyadic relevant algebra **A** with generators P is a tuple $\mathfrak{M} = \langle \mathfrak{F}^{\mathbf{A}}, v \rangle$ where $\mathfrak{F}$ is a PRA-frame and $v : P \longrightarrow \wp(W)^{\uparrow}$, extended by:

(v1) $v(1) = N$

(v2) $v(\neg a) = \neg v(a)$

(v3) $v(a \wedge b) = v(a) \cap v(b)$

(v4) $v(a \vee b) = v(a) \cup v(b)$

(v5) $v(a \rightarrow b) = v(a) \rightarrow v(b)$

(v6) $v(a \circ b) = v(a) \circ v(b)$

(v7) $v(\forall_J a) = \forall_J v(a)$

(v8) $v(\exists_J a) = \exists_J v(a)$

(v9) $v(S_\tau a) = S_\tau v(a)$

where the operations on the right-hand-side are defined by: (NEXT SLIDE)

PRA models II: 'Lifted' Operations

$$\neg X = \{\alpha \in W \mid \alpha^* \notin X\}$$
$$X \to Y = \{\alpha \in W \mid \forall \beta, \gamma \in W(R\alpha\beta\gamma \text{ and } \beta \in X \text{ imply } \gamma \in Y)\}$$
$$X \circ Y = \{\alpha \in W \mid \exists \beta, \gamma \in W(R\beta\gamma\alpha \text{ and } \beta \in X \text{ and } \gamma \in Y)\}$$
$$\forall_J X = \{\alpha \in W \mid \forall \beta(R_J\alpha\beta \text{ implies } \beta \in X)\}$$
$$\exists_J X = \{\alpha \in W \mid \exists \beta(R_J\beta\alpha \text{ and } \beta \in X)\}$$
$$S_\tau X = \{\alpha \in W \mid \exists \beta(R_{S_\tau}\beta\alpha \text{ and } \beta \in X)\}$$

and where the following (NEXT SLIDE) model conditions are satisfied.

PRA models III: Model Conditions

Where $a = \forall_J a = \exists_J a$:

(v1) $\alpha \in v(a)$ and $R_J \alpha\beta$ imply $\beta \in v(a)$

(v2) $\alpha \in v(a)$ and $R_J \beta\alpha$ imply $\beta \in v(a)$

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For $a, b \in \mathbf{A}$, we say that the equation $a = b$ is *true in a model* $\mathfrak{M}$ whenever $v(a) = v(b)$ in the model, that $a = b$ is *valid on a frame* when it is true in every model on that frame, and that $a = b$ is *valid on a class of frames* whenever it is valid on every frame in the class.

Representation Theorem (Soundness)

Theorem

Let $\mathbf{A} = \langle A, \wedge, \vee, \circ, \rightarrow, \neg, 1, \langle \forall_J, \exists_J \mid J \subseteq_\omega I \rangle, \langle S_\sigma \mid \sigma \in I^{(I)} \rangle \rangle$ be an arbitrary but fixed polyadic relevant algebra. If $\mathfrak{M}$ is a model for $\mathbf{A}$ with frame $\mathfrak{F}$, then the equations characterizing $\mathbf{A}$ *as a polyadic relevant algebra* are true in $\mathfrak{M}$.

Canonical Frame

Definition (Canonical Frames)

Let $\mathbf{A} = \langle A, \wedge, \vee, \circ, \rightarrow, \neg, 1, \langle \forall_J, \exists_J \mid J \subseteq_\omega I \rangle, \langle S_\sigma \mid \sigma \in I^{(I)} \rangle \rangle$ be a polyadic relevant algebra. The canonical frame for $\mathbf{A}$ is given as

$$\langle W^{\mathrm{c}}, N^{\mathrm{c}}, R^{\mathrm{c}}_{\circ}, *^{\mathrm{c}}, \{R^{\mathrm{c}}_{S(\tau)}\}_{\tau \in I^{(I)}}, \{R^{\mathrm{c}}_J\}_{J \subseteq_\omega I} \rangle$$

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1. W^{c} is the set of prime filters on $\mathbf{A}$;
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3. $R^{\mathrm{c}}_{\circ}\alpha\beta\gamma$ iff $\forall a, b(a \in \alpha$ and $b \in \beta$ imply $a \circ b \in \gamma)$;
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6. $R^c_J \beta\alpha$ iff $\forall a(a \in \beta$ implies $\exists_J a \in \alpha)$;

Key Squeeze Lemma

Lemma (Polyadic Squeeze)

Let $\mathbf{A} = \langle A, \wedge, \vee, \circ, \to, \neg, 1, \langle \forall_J, \exists_J \mid J \subseteq_\omega I \rangle, \langle S_\sigma \mid \sigma \in I^{(I)} \rangle \rangle$ be a polyadic relevant algebra.

(i) If $R'_\circ \beta\gamma\alpha$ and α is a prime theory, then there are prime theories β' and γ' where $\beta \subseteq \beta'$, $\gamma \subseteq \gamma'$, and $R^c_\circ \beta'\gamma'\alpha$.

(ii) If $R'_{S_\tau} \beta\alpha$ and α is a prime theory, then the set $\gamma = \{A \mid S_\tau A \in \alpha\}$ is a prime theory and $R'_{S_\tau} \hat{\gamma}\alpha$.

(iii) $R'_J \beta\alpha$ and and α is a prime theory, then there is a prime theory β' where $\beta \subseteq \beta'$ and $R^c_J \beta\alpha$

Embedding Theorem (Completeness)

Theorem

Let $\mathbf{A} = \langle A, \wedge, \vee, \circ, \rightarrow, \neg, 1, \langle \forall_J, \exists_J \mid J \subseteq_\omega I \rangle, \langle S_\sigma \mid \sigma \in I^{(I)} \rangle \rangle$ be an arbitrary but fixed polyadic relevant algebra. Then there is a model $\mathfrak{M}$ on frame $\mathfrak{F}$ for $\mathbf{A}$ such that the algebra of sets of $\mathfrak{F}$ is isomorphic to $\mathbf{A}$. That is, there is a valuation v_c that is an injective homomorphism from $\mathbf{A}$ into the algebra of sets of $\mathfrak{F}$: $v_c(a) = v_c(b)$ iff $a = b$ in $\mathbf{A}$.

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Sketchiest of Sketches.

Let $v_c(a) = \{\alpha \mid a \in \alpha \text{ and } \alpha \text{ is a prime filter on } \mathbf{A}\}$ $\square$

Fine's Incompeteness Result

Fine's Incompleteness

The formula

$$[(P \rightarrow \exists x Ex) \wedge \forall x((P \rightarrow Fx) \vee (Gx \rightarrow Hx))] \rightarrow$$

$$\rightarrow \{[\forall x((Ex \wedge Fx) \rightarrow Q) \vee \forall x((Ex \rightarrow Q) \vee Gx)] \rightarrow [\exists x Hx \vee (P \rightarrow Q)]\}$$

is valid in the *obvious constant domain semantics*, but not a theorem of **RQ**.

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Fine's Incompleteness

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$$[(P \to \exists x Ex) \land \forall x((P \to Fx) \lor (Gx \to Hx))] \to$$

$$\to \{[\forall x((Ex \land Fx) \to Q) \lor \forall x((Ex \to Q) \lor Gx)] \to [\exists x Hx \lor (P \to Q)]\}$$

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1. (Apparently,) according to Fine: the formula appears naturally in the process of trying to prove completeness. (We find this to be false...)
2. The problem appears more general: proving completeness for first-order relevant logics weaker than those for which Fine's result applies appear problematical
   2.1 Similar to the case of classical first-order modal logic
   2.2 Tension between (1) witnessing a theory and (2) ensuring counterexamples to false 'modal' operators (here, the conditional)

Why Fine's Incompleteness result? (WIP)

Conjecture

The polyadic relevant algebras do not admit the residuals of every substitution operator, and there is frame incompleteness analogous to the first-order modal classical logic.

THANKS!

Kit Fine.
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