Knowledge on a Budget

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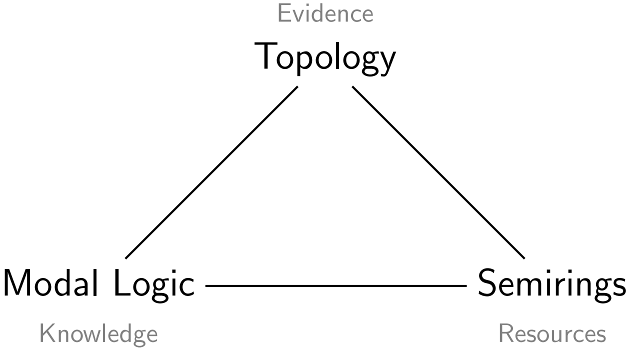
Evidence
Topology
Modal Logic
Semirings
Knowledge
Resources

Background: Topological Evidence Logic

Topological space: $\langle X, \mathcal{T} \rangle$ with $\mathcal{T} \subseteq \mathcal{P}(X)$ closed under finite intersections and arbitrary unions (in particular $\varnothing, X \in \mathcal{T}$)

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- Topological Evidence Logic (TEL): $P \subseteq X$ is *justified* iff $U \subseteq P$ for some *dense* $U \in \mathcal{T}$. It is *known* at x if additionally $x \in U$. [Baltag et al. 2016/2022, Özgün 2017]

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Our Goal: Expand `TEL` with a model of *resource constraints*.

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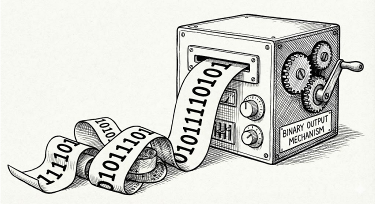
Outline of this talk:

- Semantics via semiring-annotated topological spaces (seats)
- Completeness for seat-based logics
- Expansions with global modality, weighted knowledge operators
- Some sufficient conditions for decidability

Example:
Observing Binary Streams



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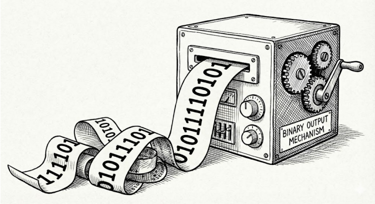

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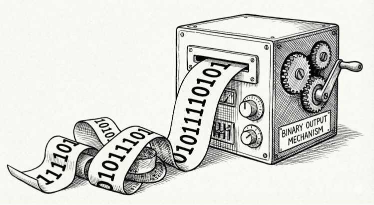

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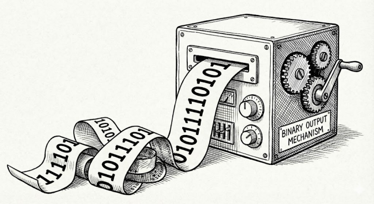

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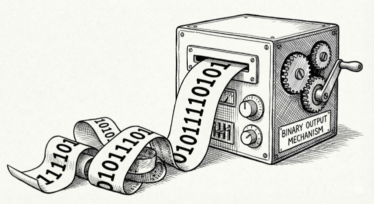

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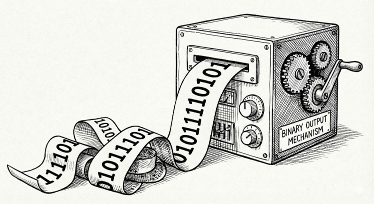

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(i) if $n \to U$, then $n + m \to U$
(ii) if $n \to U$ and $U \subseteq V$, then $n \to V$
(iii) if $n, m \to U$, then $\min(n, m) \to U$
(iv) if $n \to U$ and $m \to V$, then $n + m \to U \cap V$

Resource Semirings

Semiring $\langle K, \oplus, \odot, \mathbb{0}, \mathbb{1} \rangle$:

$\langle K, \odot, \mathbb{1} \rangle$ monoid (resource combination)

$\langle K, \oplus, \mathbb{0} \rangle$ commutative monoid (resource choice)

$$a(b \oplus c) = ab \oplus ac \qquad (a \oplus b)c = ac \oplus bc \qquad \mathbb{0}a = \mathbb{0} = a\mathbb{0}$$

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- $a \oplus b$ ‘*a or b*’
- $\mathbb{0}$ ‘*strongest resource*’ $(a \oplus \mathbb{0} = a)$

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Examples

- Tropical semiring $\langle \mathbb{N} \cup \{\infty\}, \min, +, \infty, 0 \rangle$ (discrete resources)
- Viterbi semiring $\langle [0,1], \max, \cdot, 0, 1 \rangle$ (probabilities)
- Łukasiewicz semiring $\langle [0,1], \max, \odot, 0, 1 \rangle$ (truth degrees)

Properties of 'Sufficiency'

Fix $\langle X, \mathcal{T} \rangle$ and semiring K. Read $a \xrightarrow{x} U$ as '*resource* $a \in K$ *is sufficient to obtain evidence* $U \in \mathcal{T}$ *in state* $x \in X$'.

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(v) *Available tautologies:* $\mathbb{0} \xrightarrow{x} X$
(tautological evidence X can be obtained)

Semiring-Annotated Topological Spaces (seats)

Seat: $\mathbf{S} = \langle X, \mathcal{T}, \mathcal{A}_K \rangle$ where
$\langle X, \mathcal{T} \rangle$ is a topological space, $K = \langle K, \oplus, \odot, \mathbb{0}, \mathbb{1} \rangle$ is a semiring and

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satisfies the following (for all $a \in K$, all $U, V \in \mathcal{T}$, and all $x \in X$):

$$a \in \mathcal{A}_K(U, x) \;\;\&\;\; b \in K \implies ab,\, ba \in \mathcal{A}_K(U, x) \tag{1}$$

$$U \subseteq V \implies \mathcal{A}_K(U, x) \subseteq \mathcal{A}_K(V, x) \tag{2}$$

$$a, b \in \mathcal{A}_K(U, x) \implies a \oplus b \in \mathcal{A}_K(U, x) \tag{3}$$

$$a \in \mathcal{A}_K(U, x) \;\;\&\;\; b \in \mathcal{A}_K(V, x) \implies ab \in \mathcal{A}_K(U \cap V, x) \tag{4}$$

$$\mathbb{0} \in \mathcal{A}_K(X, x) \tag{5}$$

Language and Models

Language: For each countable semiring K, define ($p \in Prop$, $a \in K$)

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \mathrm{F}_a\varphi$$

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Model: $\mathbf{M} = \langle X, \mathcal{T}, \mathcal{A}_K, \mathcal{V} \rangle$ where $\mathcal{V}\colon Prop \to \mathcal{P}(X)$.

- $\mathbf{M}, x \models \Box\varphi$ iff $\exists U\colon x \in U \in \mathcal{T}$ & $U \subseteq [\![\varphi]\!]$
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$\Box\varphi$ — '*there is factive evidence for* φ'

$\mathrm{F}_a\varphi$ — '*there is evidence for* φ *available for* $a \in K$'

$\Box_a\varphi := \Box\varphi \wedge \mathrm{F}_a\varphi$ — '*there is factive evidence for* φ *available for* a'

Logic of all K-Seats

$\mathbf{S4}_K$ is the extension of **S4** with:

$$\mathrm{F}_a\varphi \to (\mathrm{F}_{ab}\varphi \wedge \mathrm{F}_{ba}\varphi) \tag{A1}$$

$$\mathrm{F}_a\varphi \to \mathrm{F}_a\Box\varphi \tag{A2}$$

$$\mathrm{F}_a\varphi \wedge \mathrm{F}_b\psi \to \mathrm{F}_{a\oplus b}(\Box\varphi \vee \Box\psi) \tag{A3}$$

$$\mathrm{F}_a\varphi \wedge \mathrm{F}_b\psi \to \mathrm{F}_{ab}(\varphi \wedge \psi) \tag{A4}$$

$$\mathrm{F}_{\mathbb{0}}\top \tag{A5}$$

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Theorem (Strong Completeness)

$\Gamma \vdash_{\mathbf{S4}_K} \varphi$ iff $\Gamma \models \varphi$ is valid on all K-seats.

Logics of special K-seats

We call say the seat $\langle X, \mathcal{T}, \mathcal{A}_K \rangle$ is

- *bounded*: $\mathbb{1} \in \mathcal{A}_K(X, x)$ and $\mathbb{0} \in \mathcal{A}_K(\emptyset, x)$
- *uniform*: $\mathcal{A}_K(U, x) = \mathcal{A}_K(U, y)$
- *strong*: $a \oplus b \in \mathcal{A}_K(U, x) \Rightarrow a, b \in \mathcal{A}_K(U, x)$

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$$\mathrm{F}_{\mathbb{1}} \top \wedge \mathrm{F}_{\mathbb{0}} \bot \tag{Ab}$$

$$(\mathrm{F}_a \varphi \to \mathrm{F}_{\mathbb{1}} \mathrm{F}_a \varphi \wedge \Box \mathrm{F}_a \varphi) \wedge (\neg \mathrm{F}_a \varphi \to \mathrm{F}_{\mathbb{1}} \neg \mathrm{F}_a \varphi \wedge \Box \neg \mathrm{F}_a \varphi) \tag{Au}$$

$$\mathrm{F}_{a \oplus b} \varphi \to \mathrm{F}_a \varphi \wedge \mathrm{F}_b \varphi \tag{As}$$

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Theorem (Strong Completeness)

$\Gamma \vdash_{\mathbf{S4}_K^{\mathbf{b}}} \varphi$ iff $\Gamma \models \varphi$ is valid on all *b*-seats.
$\Gamma \vdash_{\mathbf{S4}_K^{\mathbf{bu}}} \varphi$ iff $\Gamma \models \varphi$ is valid on all *bu*-seats.
$\Gamma \vdash_{\mathbf{S4}_K^{\mathbf{bus}}} \varphi$ iff $\Gamma \models \varphi$ is valid on all *bus*-seats.

Completeness: Proof Sketch

Canonical Model $\mathbf{M}^{\mathbf{S4}_K} = \langle X, \mathcal{T}, \mathcal{A}_K, \mathcal{V} \rangle$, where

- X is the set of all maximal $\mathbf{S4}_K$-consistent theories $\Gamma \subseteq \mathfrak{L}_K$
  we write $|\varphi| := \{\Gamma \in X \mid \varphi \in \Gamma\}$
- $\mathcal{T}$ is the topology generated by the basis $\{|\Box\varphi| \mid \varphi \in \mathfrak{L}_K\}$
  (standard for topo semantics e.g. [Aiello et al. 2003])
- $\mathcal{A}_K(U, \Gamma) = \{a \in K \mid \exists\varphi \colon \mathrm{F}_a\varphi \in \Gamma \;\;\&\;\; |\Box\varphi| \subseteq U\}$;
- $\mathcal{V}(p) = |p|$.

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- $\mathcal{A}_K(U, \Gamma) = \{a \in K \mid \exists\varphi \colon \mathrm{F}_a\varphi \in \Gamma \;\; \& \;\; |\Box\varphi| \subseteq U\}$;
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This is a seat (use axioms A1-A5) and Truth Lemma $[\![\varphi]\!] = |\varphi|$.
Bounded and strong go through easily via corresponding axioms.

Completeness: Proof Sketch

Canonical Model $\mathbf{M}^{\mathbf{S4}_K} = \langle X, \mathcal{T}, \mathcal{A}_K, \mathcal{V} \rangle$, where

- X is the set of all maximal $\mathbf{S4}_K$-consistent theories $\Gamma \subseteq \mathfrak{L}_K$
  we write $|\varphi| := \{\Gamma \in X \mid \varphi \in \Gamma\}$
- $\mathcal{T}$ is the topology generated by the basis $\{|\Box\varphi| \mid \varphi \in \mathfrak{L}_K\}$
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Uniformity is more tricky. We *anchor* the canonical model in m.c.s. Λ:
$X^\Lambda =$ the set of $\Gamma \in X$ that agree with Λ on all $\mathrm{F}_a\varphi$

Motivation: Expansions with Global Modality

Expanded Language by *global modality* A

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \mathrm{F}_a\varphi \mid \mathrm{A}\varphi$$

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$$\mathrm{B}_a^b\varphi := \mathrm{A}\Diamond_a\Box_b\varphi$$

('*there is b-evidence for* φ *consistent with all a-evidence*')

$$\mathrm{K}_a^b\varphi := \Box_b\varphi \wedge \mathrm{B}_a^b\varphi$$

('*there is truthful b-evidence for* φ *consistent with all a-evidence*')

Interesting Notions

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  ('φ *can be justified superficially but there is no factive evidence for it*')
- Bias: $\mathrm{B}_a^\epsilon\varphi$
  ('*evidence for* φ *is easily accessible, but it survives attacks of level a*')

Logic: Expansion with Global Modality

The logic $\mathbf{S4}^{\mathbf{bus}}_{K\forall}$ is the extension of $\mathbf{S4}^{\mathbf{bus}}_{K}$ with

$$\mathrm{A}\varphi \to \varphi \tag{$\forall$1}$$

$$\mathrm{A}\varphi \to \mathrm{A}\mathrm{A}\varphi \tag{$\forall$2}$$

$$\varphi \to \mathrm{A}\mathrm{E}\varphi \tag{$\forall$3}$$

$$\mathrm{A}\varphi \wedge \mathrm{A}\psi \to \mathrm{A}(\varphi \wedge \psi) \tag{$\forall$4}$$

$$\mathrm{A}\varphi \to \Box\varphi \wedge \mathrm{F}_{\mathbb{1}}\varphi \tag{$\forall$5}$$

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Theorem (Strong Completeness)

$\Gamma \vdash_{\mathbf{S4}^{\mathbf{bus}}_{K\forall}} \varphi$ iff $\Gamma \models \varphi$ is valid on all *bus*-seats.

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Theorem (Decidability)

$\mathbf{S4}_K$, $\mathbf{S4}^{\mathbf{b}}_K$, $\mathbf{S4}^{\mathbf{bu}}_K$ are decidable for commutative ideally checkable K.

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$$\mathcal{A}_K^\Sigma(V, \bar{x}) = \mathcal{A}_K(\pi^{-1}(V), r(\bar{x}))$$

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Step 2: Show it suffices to check finitely many 'Σ-quasi models'
These generate enough relevant models for checking satisfiability

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- Decidability of $\mathbf{S4}_K$, $\mathbf{S4}_K^{\mathbf{b}}$, $\mathbf{S4}_K^{\mathbf{bu}}$ for ideally checkable K

Future Research

- More (un)decidability and complexity

Decidability for expansions with global modality, complexity relative to K

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Thanks for your attention!

Preprint available here.

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