Ultracontact algebras and stack systems

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Outline

Boolean contact algebras

A Boolean contact algebra is a structure $\mathfrak{B} := \langle B, \mathbf{C} \rangle$ such that:

1. B is a Boolean algebra, elements of B are called regions.
2. $\mathbf{C} \subseteq B \times B$ is a contact relation satisfying:

$$0 \not\mathbf{C}\, x \tag{C0}$$

$$x \neq 0 \rightarrow x \,\mathbf{C}\, x \tag{C1}$$

$$x \,\mathbf{C}\, y \rightarrow y \,\mathbf{C}\, x \tag{C2}$$

$$x \,\mathbf{C}\, y \wedge y \leqslant z \rightarrow x \,\mathbf{C}\, z \tag{C3}$$

$$x \,\mathbf{C}\, (y + z) \rightarrow x \,\mathbf{C}\, y \vee x \,\mathbf{C}\, z\,. \tag{C4}$$

Boolean contact algebras: the canonical algebras

The canonical interpretation of BCAs is obtained by taking a Boolean algebra whose regions are regular closed (or regular open) sets of a topological space, and defining:

$$x \mathbf{C}_\tau y \;:\longleftrightarrow\; x \cap y \neq \emptyset \quad \text{for r.c. sets}$$
$$x \mathbf{C}_\tau y \;:\longleftrightarrow\; \operatorname{Cl} x \cap \operatorname{Cl} y \neq \emptyset \quad \text{for r.o. sets.}$$

In other words, we take pairs $\mathfrak{B} := \langle S, \mathbf{C}_\tau \rangle$ where $S \leq \mathrm{RC}(X)$ for a topological space $\langle X, \tau \rangle$.

Important: Above, $\cap$ is not the meet operation of $\mathrm{RC}(X)$, the latter is given by

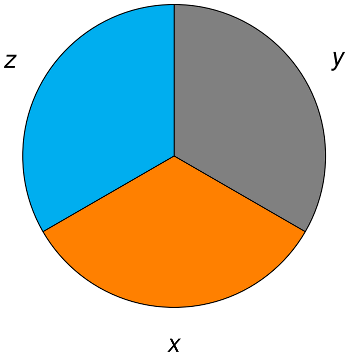
$$x \cdot y = \operatorname{Cl} \operatorname{Int}(x \cap y).$$

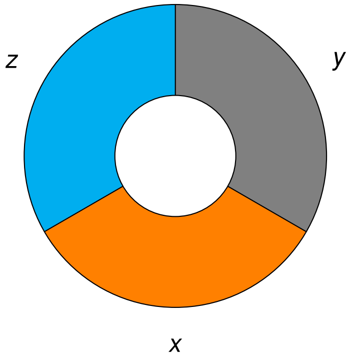
Theorem (Representation Theorem)

Every BCA $\mathfrak{B} := \langle B, \mathbf{C} \rangle$ can be embedded into the algebra $\langle RC(X), \tau \rangle$ of a T_0, semi-regular, and compact space $\langle X, \tau \rangle$.

Boolean contact algebras: a limitation

Since we have a formal theory of the binary relation of nearness for regions in space, we can think of a more general phenomenon in which collections (including infinite ones) of regions are close to each other. The two situations below are the same from the point of view of what can be said about them in terms of the contact relation: any pair of regions is in contact. Yet these are different situations.





Hypercontact algebras

Definition (Lipparini, 2025)

A pair $\langle B, \Delta \rangle$ is a hypercontact algebra if B is a Boolean algebra, and Δ is a collection of finite subsets of B such that

(H1) If $0 \in F$, then $F \notin \Delta$.

(H2) If $x \neq 0$, then $\{x\} \in \Delta$.

(H3) If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

(H4) If $F \in \Delta$ and there is an $x \in F$ such $x \leq y$, then $F \cup \{y\} \in \Delta$.

(H5) If $F \cup \{x + y\} \in \Delta$, then $F \cup \{x\} \in \Delta$ or $F \cup \{y\} \in \Delta$.

Our aim: a generalization to arbitrary collections (not only finite ones).

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Ultracontact algebras: the support relation

Definition

For $F, G \in 2^B$, we define $F \preceq G$ (in words: F supports G) iff for every $g \in G$ there is $f \in F$ such that $f \leq g$ (in BA parlance, F is dense in G).

Proposition

$F \preceq G$ if and only if $G \subseteq \uparrow F$.

Example

Any set $F \subseteq B$ supports each subset of itself.

Ultracontact algebra: the definition

Definition (Carai et al., 2026)

An ultracontact algebra is a pair $\langle B, \mathscr{K} \rangle$ such that B is a Boolean algebra and $\mathscr{K} \subseteq 2^B$ is a family that satisfies the following axioms for all $F, G \subseteq B$:

(K0) $\emptyset \notin \mathscr{K}$.

(K1) If $0 \in F$, then $F \notin \mathscr{K}$.

(K2) If $x \neq 0$, then $\{x\} \in \mathscr{K}$.

(K3) If $F \leq G \neq \emptyset$ and $F \in \mathscr{K}$, then $G \in \mathscr{K}$.

(K4) If $F + G \in \mathscr{K}$, then $F \in \mathscr{K}$ or $G \in \mathscr{K}$,

where $+$ is a complex sum (or Minkowski sum)

$$F + G := \{f + g \mid f \in F \wedge g \in G\}.$$

Ultracontact algebras: example in a regular closed algebra

Example

Let X be a topological space, $\mathrm{RC}(X)$ be its complete BA of regular closed sets, and let $S \leq \mathrm{RC}(X)$. For families $\mathcal{F}, \mathcal{G} \subseteq 2^{\mathrm{RC}(X)}$ we define

$$\mathcal{F} \uplus \mathcal{G} := \{F \cup G \mid F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}.$$

Let $\mathscr{K}^S \subseteq 2^S$ be the family of all non-empty $\mathcal{F} \in 2^S$ such that $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$, where $\bigcap$ is the set-theoretical intersection (which is the infimum operation of 2^X, but not necessarily the infimum of $\mathrm{RC}(X)$ or S). Then, $\mathscr{K}^S$ meets the conditions for ultracontact:

(TK0) $\emptyset \notin \mathscr{K}^S$.

(TK1) If $\emptyset \in \mathcal{F}$, then $\mathcal{F} \notin \mathscr{K}^S$.

(TK2) If $F \neq \emptyset$, then $\{F\} \in \mathscr{K}^S$.

(TK3) If $\mathcal{F} \leq \mathcal{G} \neq \emptyset$ and $\mathcal{F} \in \mathscr{K}^S$, then $\mathcal{G} \in \mathscr{K}^S$.

(TK4) If $\mathcal{F} \uplus \mathcal{G} \in \mathscr{K}^S$, then $\mathcal{F} \in \mathscr{K}^S$ or $\mathcal{G} \in \mathscr{K}^S$.

The lattice of ultracontacts

Every Boolean algebra carries at least two ultracontacts:

$$\mathscr{K}_{\mathsf{min}} := \{M \in 2^B_+ \mid (\exists x \in B^+)\, x \leq M\}$$

and

$$\mathscr{K}_{\mathsf{max}} := \{M \in 2^B_+ \mid 0 \notin M\}.$$

Theorem

For any ultracontact $\mathscr{K}$ on an algebra B, $\mathscr{K}_{\mathsf{min}} \subseteq \mathscr{K} \subseteq \mathscr{K}_{\mathsf{max}}$.

Theorem

$\mathbf{UC}(B)$ *is a complete lattice, where joins are given by set-theoretical unions. Moreover, it satisfies infinite meet distributivity, i.e.,*

$$\mathscr{K} \cup \bigwedge_{i \in I} \mathscr{K}_i = \bigwedge_{i \in I} (\mathscr{K} \cup \mathscr{K}_i)$$

so $\mathbf{UC}(B)$ *is co-frame.*

Outline

Extending ultracontacts

If $\mathscr{K}$ is an ultracontact on B and $M \subseteq B$, let

$$\mathscr{K}^{M} := \mathscr{K} \cup \left\{ F \in 2^{B}_{+} \,\middle|\, F \subseteq M \right\}.$$

If $\mathcal{M}$ is a collection of subsets of B, we put

$$\mathscr{K}^{\mathcal{M}} := \bigcup \left\{ \mathscr{K}^{M} \,\middle|\, M \in \mathcal{M} \right\}.$$

Extending ultracontacts: grills

Definition (Choquet, 1947)

A proper non-empty subset G of B is called a grill, if it satisfies

(Gr0) $1 \in G$ and $0 \notin G$.

(Gr1) If $a \in G$ and $a \leq b$, then $b \in G$.

(Gr2) If $a + b \in G$, then either $a \in G$ or $b \in G$.

Grills are precisely complements of proper ideals.

Theorem

A subset G of a Boolean algebra B is a grill if G is a union of a non-empty family of ultrafilters. Also, if F is a filter and G is a grill such that $F \subseteq G$, then there exists an ultrafilter $F \subseteq U \subseteq G$.

Grills are the building blocks of representations of BCAs: Düntsch and Winter (2005); Dimov and Vakarelov (2006).

Extending ultracontacts

Theorem

If G is a grill, then $\mathscr{K}^G := \mathscr{K} \cup \{M \neq \emptyset \mid M \subseteq G\}$ is the smallest ultracontact expanding $\mathscr{K}$ and containing G.

Theorem

If $\mathscr{K}$ is an ultracontact on B and $\emptyset \neq M \notin \mathscr{K}$, then: $\mathscr{K}^M$ is an ultracontact iff M is a grill.

(1) $\mathscr{K}^M$ is a method of expanding an ultracontact $\mathscr{K}$ with all non-empty subsets of a set M not in $\mathscr{K}$.

(2) The method works iff M is a grill.

Expanding ultracontacts

Example

Let $B := 2^{\mathbb{N}}$. $\mathscr{K}_{\text{min}} \subseteq 2_{+}^{(2^{\mathbb{N}})}$ is the family of those families of subsets of $\mathbb{N}$ that have a non-empty intersection: $\mathscr{A} \in \mathscr{K}_{\text{min}}$ iff $\bigcap \mathscr{A} \neq \emptyset$.

(1) No two-element set of different singletons is in $\mathscr{K}_{\text{min}}$: $\{\{m\}, \{n\}\} \notin \mathscr{K}_{\text{min}}$, for $m \neq n$.

(2) Supoose we want to put $\{0\}$ and $\{1\}$ to be in ultracontact, so we need to add $\{\{0\}, \{1\}\}$ to $\mathscr{K}_{\text{min}}$.

(3) Yet, this is not enough, since we need to add all sets supported by $\{\{0\}, \{1\}\}$, e.g., $\{\{0, 2\}, \{1, 3\}\}$ and so on. That is, we add all non-empty subsets of $\uparrow\{0\} \cup \uparrow\{1\}$, which is a grill, as the union of two principal ultrafilters.

Expanding ultracontacts

Example

Let $B := 2^{\mathbb{N}}$. $\mathscr{K}_{\min} \subseteq 2_{+}^{(2^{\mathbb{N}})}$ is the family of those families of subsets of $\mathbb{N}$ that have a non-empty intersection: $\mathscr{A} \in \mathscr{K}_{\min}$ iff $\bigcap \mathscr{A} \neq \emptyset$.

(1) For any non-principal ultrafilter U of $2^{\mathbb{N}}$, $\mathscr{K}_{\min}^{U}$ is an ultracontact.

(2) In consequence, all co-finite sets are in ultracontact, and so are all sets in U.

(3) However, the sets $\mathbb{E}$ and $\mathbb{O}$ are not in ultracontact, as one of them is not in U, so

$$\{\mathbb{E}, \mathbb{O}\} \nsubseteq U, \qquad \text{and thus} \qquad \{\mathbb{E}, \mathbb{O}\} \notin \mathscr{K}_{\min}^{U}.$$

(4) We can put these sets into ultracontact (with many others), by taking two non-principal ultrafilters U_1 and U_2, one of them with $\mathbb{E}$, the other with $\mathbb{O}$ and expanding $\mathscr{K}_{\min}$ with all non-empty subsets of the grill $G := U_1 \cup U_2$.

Expanding ultracontacts

Example

Let $B \coloneqq 2^{\mathbb{N}}$. $\mathscr{K}_{\mathrm{min}} \subseteq 2_+^{(2^{\mathbb{N}})}$ is the family of those families of subsets of $\mathbb{N}$ that have a non-empty intersection: $\mathscr{A} \in \mathscr{K}_{\mathrm{min}}$ iff $\bigcap \mathscr{A} \neq \emptyset$.

(1) Suppose you want to make sure that any non-empty family of infinite sets is such that all its elements are in ultracontact.

(2) The family G of all infinite subsets of $\mathbb{N}$ is a grill, so $\mathscr{K}^{G}_{\mathrm{min}}$ is an ultracontact.

Outline

Contact relation from ultracontact

Each ultracontact algebra $\langle B, \mathscr{K} \rangle$ carries enough information to give rise to the contact relation determined by $\mathscr{K}$. To this end, for a $\langle B, \mathscr{K} \rangle$ we define

$$x \mathrel{\mathbf{C}_{\mathscr{K}}} y \;:\longleftrightarrow\; \{x, y\} \in \mathscr{K} .$$

$$0 \mathrel{\not\mathbf{C}} x \tag{C0}$$

$$x \neq 0 \rightarrow x \mathrel{\mathbf{C}} x \tag{C1}$$

$$x \mathrel{\mathbf{C}} y \rightarrow y \mathrel{\mathbf{C}} x \tag{C2}$$

$$x \mathrel{\mathbf{C}} y \wedge y \leqslant z \rightarrow x \mathrel{\mathbf{C}} z \tag{C3}$$

$$x \mathrel{\mathbf{C}} (y + z) \rightarrow x \mathrel{\mathbf{C}} y \vee x \mathrel{\mathbf{C}} z \,. \tag{C4}$$

Ultracontact from contact relation, and back

If $\mathbf{C}$ is a contact relation on B, then there is an ultracontact $\mathscr{K}$ such that $\mathbf{C} = \mathbf{C}_{\mathscr{K}}$.

Definition

Let $\langle B, \mathbf{C}\rangle$ be a BCA. A grill Γ is a clan if for all $\Gamma \times \Gamma \subseteq \mathbf{C}$.

$$\mathsf{Clan}(B, \mathbf{C}) := \text{all clans of } \langle B, \mathbf{C}\rangle.$$

Theorem

If $\langle B, \mathbf{C}\rangle$ is a BCA, then

$$\mathscr{K} := \mathscr{K}_{\mathsf{min}}^{\mathsf{Clan}(B,\mathbf{C})} = \bigcup_{\Gamma \in \mathsf{Clan}(B,\mathbf{C})} \mathscr{K}_{\mathsf{min}}^{\Gamma}$$

is an ultracontact on B such that $\mathbf{C} = \mathbf{C}_{\mathscr{K}}$.

Important: There may be many such ultracontacts, but the one above is probably the largest one.

Hypercontact algebras from ultracontact

Similarly, each ultracontact algebra $\langle B, \mathscr{K} \rangle$ carries enough information to give rise to the hypercontact determined by $\mathscr{K}$:

$$\Delta_{\mathscr{K}} := \{F \in \mathscr{K} \mid F \text{ is finite}\} \cup \{\emptyset\}.$$

(H1) If $0 \in F$, then $F \notin \Delta$.

(H2) If $x \neq 0$, then $\{x\} \in \Delta$.

(H3) If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

(H4) If $F \in \Delta$ and there is an $x \in F$ such $x \leq y$, then $F \cup \{y\} \in \Delta$.

(H5) If $F \cup \{x + y\} \in \Delta$, then $F \cup \{x\} \in \Delta$ or $F \cup \{y\} \in \Delta$.

Outline

Ultracontact algebras and abstract simplicial complexes

Definition

An abstract simplicial complex is a pair $\langle V, \Sigma \rangle$ such that the elements of V are called vertices and Σ is a family of finite non-empty subsets of V, the elements of Σ are called simplexes and satisfy the following two conditions:

(SC1) every singleton subset of V is in Σ (for every $v \in V$, $\{v\}$ is a simplex),

(SC2) If $S \in \Sigma$ and $\emptyset \neq T \subseteq S$, then $T \in \Sigma$ (Σ is closed under non-empty subsets, i.e., every non-empty subset of a simplex is a simplex).

Let $\mathbf{S}(V)$ be the collection of all simplexes on V.

Ultracontact algebras and abstract simplicial complexes

Fix a finite BA B and an ultracontact $\mathscr{K}$ on B. Define $\Sigma := \mathscr{K} \cap 2_+^{\mathrm{At}(B)}$.

(SC1) If a is an atom, then $a \neq 0$, and so $\{a\} \in \Sigma$.

(SC2) If F and G are non-empty sets of atoms, $G \in \Sigma$ and $F \subseteq G$, then

  (a) $G \in \mathscr{K}$ and $G \preceq F$

  (b) so by (K3) axiom, $F \in \mathscr{K}$ and so $F \in \Sigma$.

Thus, $\langle \mathrm{At}(B), \Sigma \rangle$ is an abstract simplicial complex.

Ultracontact algebras and abstract simplicial complexes

Again, let B be a finite BA, and let $\langle \mathrm{At}(B), \Sigma \rangle$ be an abstract simplicial complex. Define

$$\mathscr{K} := \left\{ F \in 2^B_+ \,\middle|\, \text{there is } H \in \Sigma \text{ with } H \preceq F \right\}$$

(a) Any $H \in \Sigma$ is a set of atoms.

(b) $H \preceq F$, if any element $f \in F$ contains at least one atom (vertex) of the simplex H.

(c) $\mathscr{K}$ is an ultracontact on B.

Ultracontact algebras and abstract simplicial complexes

Theorem

If B is a finite algebra, then the mapping σ from all ultracontacts on B to all abstract simplicial complexes on $\mathrm{At}(B)$ such that

$$\sigma(\mathscr{K}) := \mathscr{K} \cap 2_{+}^{\mathrm{At}(B)}$$

is an order isomorphism (w.r.t. the set-theoretical inclusion).

Outline

Stack systems

Definition

We say that $\mathscr{S} \subseteq \mathbf{Stack}(B)$ is a stack system if the following axioms hold for all $U, V \in \mathbf{Stack}(B)$ and all $x \in B$:

(SS0) $\emptyset \notin \mathscr{S}$.

(SS1) $B \notin \mathscr{S}$.

(SS2) If $x \neq 0$, then $\uparrow x \in \mathscr{S}$.

(SS3) If $\emptyset \neq U \subseteq V$ and $V \in \mathscr{S}$, then $U \in \mathscr{S}$.

(SS4) If $U \cap V \in \mathscr{S}$, then $U \in \mathscr{S}$ or $V \in \mathscr{S}$.

A stack algebra is a pair $\langle B, \mathscr{S} \rangle$ such that B is a Boolean algebra and $\mathscr{S}$ is a stack system on B.

Ultracontact algebras and stack systems

For an ultracontact $\mathscr{K}$, define

$$\mathscr{S}_{\mathscr{K}} := \{\uparrow F \mid F \in \mathscr{K}\}, \tag{df $\mathscr{S}_{\mathscr{K}}$}$$

and for a stack system $\mathscr{S}$, put

$$\mathscr{K}_{\mathscr{S}} := \{F \subseteq B \mid \uparrow F \in \mathscr{S}\}. \tag{df $\mathscr{K}_{\mathscr{S}}$}$$

Lemma

Let B be a Boolean algebra.

(1) $\mathscr{S}_{\mathscr{K}}$ is a stack system for every ultracontact $\mathscr{K}$.

(2) $\mathscr{K}_{\mathscr{S}}$ is an ultracontact for every stack system $\mathscr{S}$.

Theorem

The mappings $\mathscr{K} \mapsto \mathscr{S}_{\mathscr{K}}$ and $\mathscr{S} \mapsto \mathscr{K}_{\mathscr{S}}$ are mutual inverses, that is $\mathscr{K} = \mathscr{K}_{\mathscr{S}_{\mathscr{K}}}$ and $\mathscr{S} = \mathscr{S}_{\mathscr{K}_{\mathscr{S}}}$, so for any Boolean algebra B, there is a one-to-one correspondence between ultracontacts and stack systems on B.

Ultracontact algebras: representation

Open problem: Find a representation of ultracontact algebras similar to the topological representation of Boolean contact algebras?

Thank You

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