Interpolation in Two-Dimensional Cylindric Modal Logic

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Outline

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Cylindric algebras and cylindric modal logic

Cylindric algebras are algebraic models of first-order logic [Henkin, Monk and Tarski, 1971]

Finite-dimensional cylindric algebras have modal counterparts:

$S5^n$ [Gabbay and Shehtman, 1998]

- the n-dimensional product of S5;
- the diagonal-free cylindric modal logic;
- corresponds to the substitution-free n-variable fragment.

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Why two-dimensional cylindric modal logic

NExt(S5) is very simple [Scroggs, 1951]

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- $S5^3$ is not finitely axiomatizable [Johnson, 1969]
- $S5^3$ has $2^{\aleph_0}$ many undecidable extensions without the FMP [Gabbay et al., 2003]

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Interpolation via amalgamation

We study interpolation via algebraic method and duality:

- translate interpolation into amalgamation
- use duality between finite frames and finite algebras

Main Theorem A.
There are exactly 18 extensions of $S5^2$ with the Craig interpolation property

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Modal Logic $S5^2$

Recall that the bimodal modal logic $S5 \times S5$ or $S5^2$ is the least bi-modal logic containing:

- S5-modalities $\Diamond_1$ and $\Diamond_2$ and
- the commutativity axiom: $\Diamond_1\Diamond_2 p \leftrightarrow \Diamond_2\Diamond_1 p$.

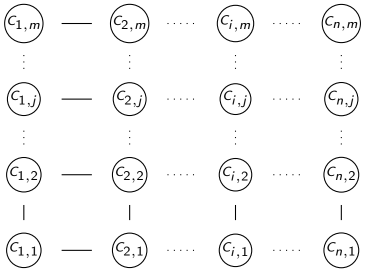
A normal extension L of $S5^2$ is a superset of $S5^2$ closed under (MP), (Nec) and (Sub)

Let NExt($S5^2$) denote the set of all normal extensions of $S5^2$

Frames and algebras for $S5^2$

An $S5^2$-frame is a triple (X, E_1, E_2) where

- E_1 and E_2 are equivalence relations on X
- $E_1 \circ E_2 = E_2 \circ E_1$



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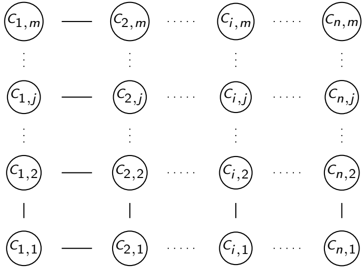
An $S5^2$-algebra is a (bi-)modal algebra $(B, \Diamond_1, \Diamond_2)$ where

- $(B, \Diamond_1)$ and $(B, \Diamond_2)$ are S5-algebras
- $\Diamond_1 \Diamond_2 a = \Diamond_2 \Diamond_1 a$ for all $a \in B$



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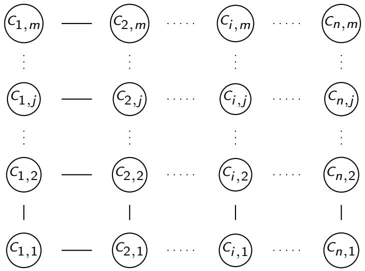

Theorem. The categories of finite $S5^2$-frames and finite $S5^2$-algebras are dually equivalent.

Interpolation

Definition. We say that a logic L has the Craig interpolation property (CIP), if for any $\overline{p}, \overline{q}, \overline{r}$ and formulas $\varphi(\overline{p}, \overline{q}), \psi(\overline{p}, \overline{r})$ with

$$\vdash_L \varphi(\overline{p}, \overline{q}) \to \psi(\overline{p}, \overline{r}),$$

then there exists $\chi(\overline{p})$ such that

$$\vdash_L \varphi(\overline{p}, \overline{q}) \to \chi(\overline{p})$$

and

$$\vdash_L \chi(\overline{p}) \to \psi(\overline{p}, \overline{r}).$$

Interpolation and Amalgamation

A class $\mathcal{V}$ of $S5^2$-algebras has the superamalgamation property if a superamalgam exists for every amalgamation triple.

Amalgamation triple:
$$\begin{tikzcd} \mathfrak{A}_2 & \\ \mathfrak{A} \arrow[u, tail, "h_2"] \arrow[r, tail, "h_1"'] & \mathfrak{A}_1 \end{tikzcd}$$

Amalgam:
$$\begin{tikzcd} \mathfrak{A}_2 \arrow[r, tail, dashed, "e_2"] & \mathfrak{B} \\ \mathfrak{A} \arrow[u, tail, "h_2"] \arrow[r, tail, "h_1"'] & \mathfrak{A}_1 \arrow[u, tail, dashed, "e_1"'] \end{tikzcd}$$

An amalgam is a superamalgam if for $\{i, j\} = \{1, 2\}$,

$$e_i(a) \leq e_j(b) \text{ implies } \exists c \in \mathfrak{A} \text{ such that } a \leq_i h_i(c) \text{ and } h_j(c) \leq_j b.$$

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Dually, we define co-superamalgamation property for classes of spaces:

$$\begin{tikzcd} X_2 \arrow[d, two heads, "f_2"'] & \\ X & X_1 \arrow[l, two heads, "f_1"] \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[d, two heads, "f_2"'] & Y \arrow[l, two heads, dashed, "g_2"'] \arrow[d, two heads, dashed, "g_1"] \\ X & X_1 \arrow[l, two heads, "f_1"] \end{tikzcd}$$

Interpolation and Amalgamation

Theorem [Maksimova, 1979]. A normal modal logic L has the Craig interpolation property iff the class of all L-algebras is superamalgamable.

Corollary. A normal modal logic L has the Craig interpolation property iff the class of all L-spaces is co-superamalgamable.

Corollary. A normal modal logic L lacks the CIP if $\mathrm{Fr}(L)$ is not co-amalgamable

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Interpolation in intuitionistic and modal logics

In the setting of superintuitionistic and modal logics, several of these classifications exist

Theorem [Maksimova, 1977] There exist exactly 8 superintuitionistic logics with the CIP

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Can we give a full classification of extensions of $S5^2$ with the interpolation property?

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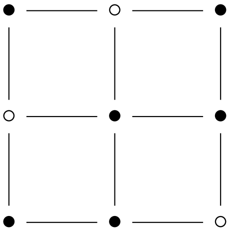
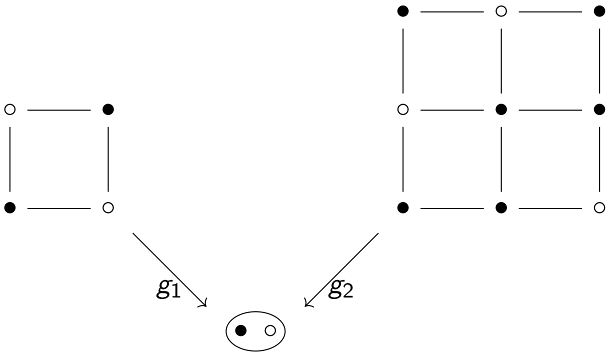


Interpolation and Amalgamation for $S5^2$

A co-amalgamation triple in Fr($S5^2$) without co-amalgam



Lemma. *L* cannot have both depth and width greater than 2

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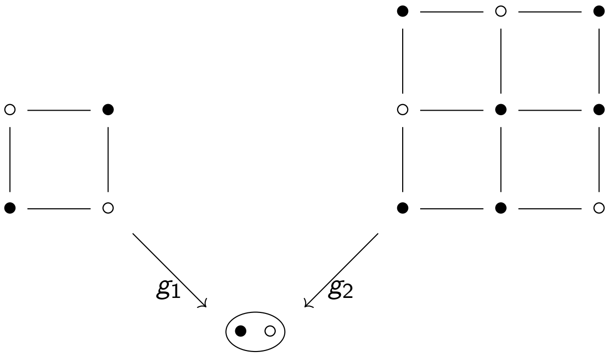

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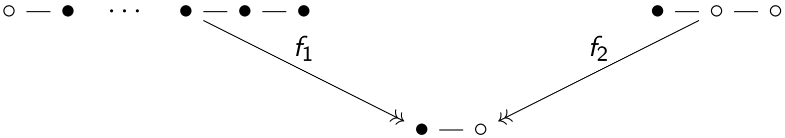
Extensions of S5^2 with the CIP

Second observation:

If L has an E_1-chain of length $n > 2$, then it has an E_1-chain of length $n + 1$



Lemma. *L cannot have finite depth (or width) greater than 2*

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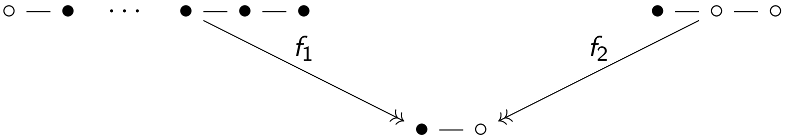
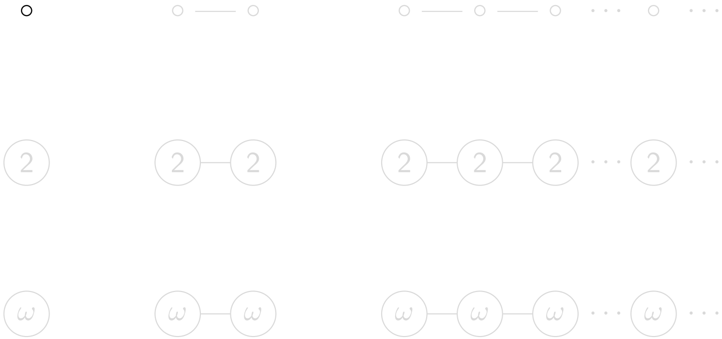


Lemma. L cannot have finite depth (or width) greater than 2

Suppose L has depth 1. Then L is the logic of one of the following frames:



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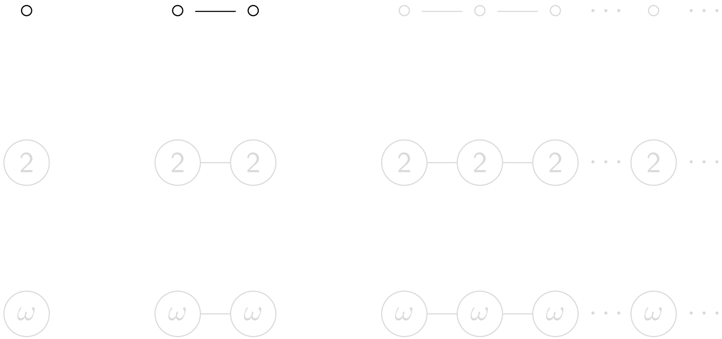

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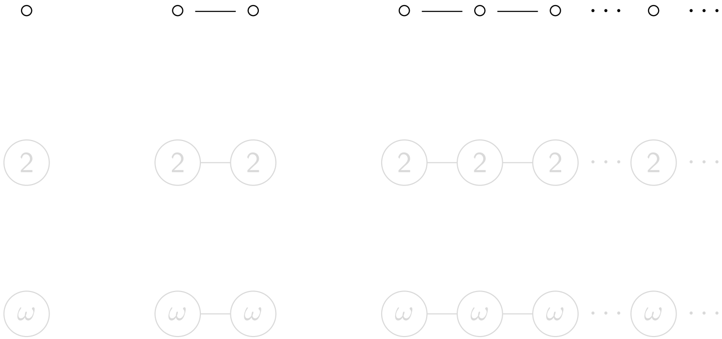

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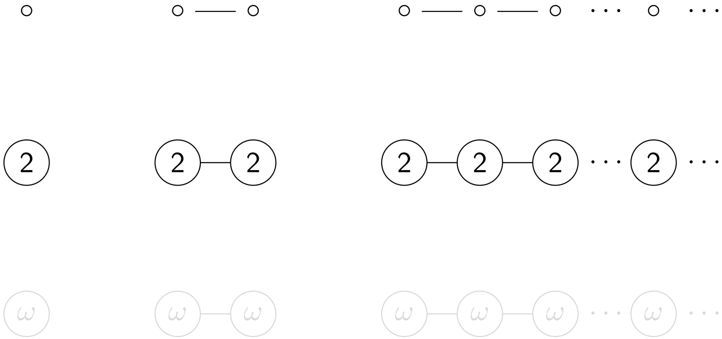

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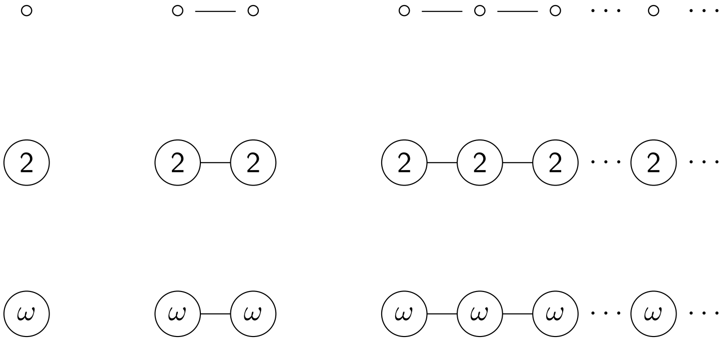
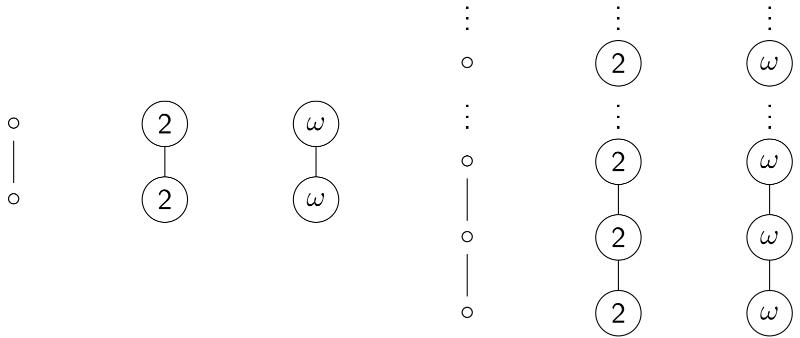
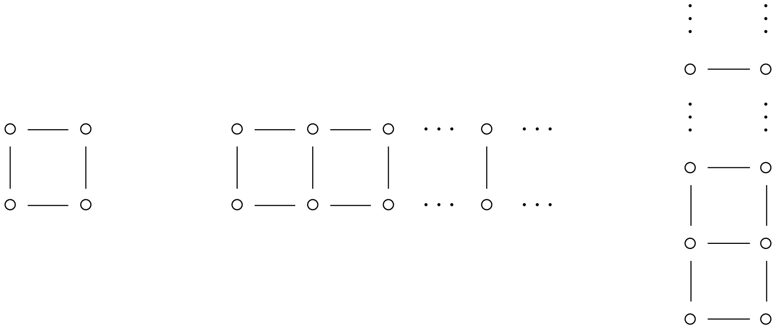


Analogously, if L has width 1 and depth > 1, then L must be the logic of one of the following:



Suppose L has $\min(\text{depth}, \text{width}) = 2$. Then L is the logic of one of the following frames:



Note that L cannot have $\min(\text{depth}, \text{width}) > 2$. We get a full characterization!

Interpolation in NExt($S5^2$)

Theorem. There exist exactly 18 normal extensions of $S5^2$ with the CIP.

Thank you!

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