Axiomatization of logics complete with respect to 3-element semi-Heytinga algebras

S. Stokłosa[1]

[1] Department of Philosophy
Jagiellonian University

Topology, Algebra, and Categories in Logic, 2026

Outline

Open Problem

In 2007 Sankappanavar, following the tradition of expanding classical logic to intuitionistic logic, has proposed a new generalization of intuistionistic logic via algebraic methods. This proposed class of algebras is called the class of semi-Heyting algebras. Along with formulating them and showing some important properties, Sankappanavar put forth some open problems. One of them, problem 14.3, calls for

> ***Problem 14.3.*** *Axiomatize the three-valued semi-Heyting logics* $\mathcal{L}_2 - \mathcal{L}_{10}$ *from the logical point of view.*

Definition (Heyting Algebra)

An algebra $\mathcal{L} = \langle L, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a Heyting algebra if the following conditions hold:

(H1) $\langle L, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a lattice with $0, 1$.

(H2) $x \wedge (x \rightarrow y) \approx x \wedge y$.

(H3) $x \wedge (y \rightarrow z) \approx x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$.

(H4) $(x \wedge y) \rightarrow x \approx 1$.

Definition (semi-Heyting Algebra)

An algebra $\mathcal{L} = \langle L, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a semi-Heyting algebra if the following conditions hold:

(SH1) $\langle L, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a lattice with $0, 1$.

(SH2) $x \wedge (x \rightarrow y) \approx x \wedge y$.

(SH3) $x \wedge (y \rightarrow z) \approx x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$.

(SH4) $x \rightarrow x \approx 1$.

For both Heyting and semi-Heyting algebras the following hold:

1. The variety is arithmetical.
2. Algebras are distributive.
3. Congruences of algebras are determined by filters and vice versa.

Hence, semi-Heyting algebras preserve some ‘nice’ properties of Heyting algebras (from perspective of algebraizable logics).

Lemma (Heyting)

- (a) $1 \to a \approx a$,
- (b) $a \leq b$ *iff* $a \leq a \to b$,
- (c) $a \leq b \to a$,
- (d) $a \to b \approx 1$ *iff* $a \leq b$,
- (e) $x \leq a \to b$ *iff* $x \wedge a \leq b$,
- (f) $0 \to 1 \approx 1$.

Lemma (semi-Heyting)

- (a) $1 \to a \approx a$,
- (b) $a \leq b$ *implies* $a \leq a \to b$,
- (c) $a \leq b$ *implies* $a \leq b \to a$,
- (d) $a \to b \approx 1$ *implies* $a \leq b$,
- (e) $x \leq a \to b$ *implies* $x \wedge a \leq b$,
- (f) $a \leq 0 \to 1$ *iff* $a \leq 0 \to a$.

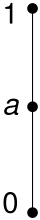
Definition

We will call $\mathcal{A} \in \mathcal{SH}$ a **semi-Heyting chain** (in symbols: $\mathcal{SH}^c$) if the lattice reduct is totally ordered.
We will call $\mathcal{A} \in \mathcal{SH}$ a **Stone semi-Heyting** (in symbols: $\mathcal{SSH}$) if $\mathcal{A} \models x^* \vee x^{**} \approx 1$. Where $x^* = x \to 0$ is a pseudo-complement of x.

Lemma

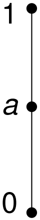
$\mathcal{SH}^c \subseteq \mathcal{SSH}$

$\mathcal{L}_1$:



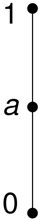
$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

$\mathcal{L}_2$:



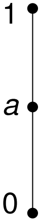
$\to$	0	a	1
0	1	a	1
a	0	1	1
1	0	a	1

$\mathcal{L}_3$:



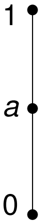
$\to$	0	a	1
0	1	1	1
a	0	1	a
1	0	a	1

$\mathcal{L}_4$:



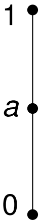
$\to$	0	a	1
0	1	a	1
a	0	1	a
1	0	a	1

$\mathcal{L}_5$:



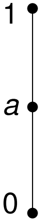
$\to$	0	a	1
0	1	a	a
a	0	1	1
1	0	a	1

$\mathcal{L}_6$:



$\to$	0	a	1
0	1	1	a
a	0	1	1
1	0	a	1

$\mathcal{L}_7$:

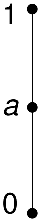


$\to$	0	a	1
0	1	a	a
a	0	1	a
1	0	a	1

$\mathcal{L}_8$:

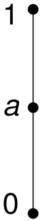
$\to$	0	a	1
0	1	1	a
a	0	1	a
1	0	a	1

$\mathcal{L}_9$:



$\to$	0	a	1
0	1	0	0
a	0	1	1
1	0	a	1

$\mathcal{L}_{10}$:



$\to$	0	a	1
0	1	0	0
a	0	1	a
1	0	a	1

Let $\mathcal{A} = (A, F_A)$ and $\mathcal{B} = (A, F_B)$ be algebras with the same underlying set A, but with possibly different signatures.

Definition (Term-equivalence)

The algebras $\mathcal{A} = (A, F_A)$ and $\mathcal{B} = (A, F_B)$ are *term-equivalent* if there are maps

$$\tau : F_A \to \mathrm{Term}(F_B), \qquad \sigma : F_B \to \mathrm{Term}(F_A),$$

such that the following holds.

1. If $f \in F_A$ is n-ary, then $\tau(f)$ is an n-ary F_B-term such that for all $a_1, \dots, a_n \in A$,
$$f^{\mathcal{A}}(a_1, \dots, a_n) = \tau(f)^{\mathcal{B}}(a_1, \dots, a_n).$$
2. If $g \in F_B$ is m-ary, then $\sigma(g)$ is an m-ary F_A-term such that for all $a_1, \dots, a_m \in A$,
$$g^{\mathcal{B}}(a_1, \dots, a_m) = \sigma(g)^{\mathcal{A}}(a_1, \dots, a_m).$$

Proposition

If $\mathcal{A}$ and $\mathcal{B}$ are term-equivalent, then congruences and universes of subalgebras of $\mathcal{A}$ and $\mathcal{B}$ are the same.

Proposition

For every F_A-quasi-equation e,

$$\mathcal{A} \models e \quad \iff \quad \mathcal{B} \models \tau(e).$$

Similarly, for every F_B-quasi-equation q,

$$\mathcal{B} \models q \quad \iff \quad \mathcal{A} \models \sigma(q).$$

Define $\delta_g = \tau(\sigma(g))(x_1, \ldots, x_n) \approx g(x_1, \ldots, x_n)$ and let $\Delta_B = \{\delta_g : g \in F_B\}$. Then:

Theorem

Let E be a basis of $\mathbb{Q}(\mathcal{A})$. Then $\tau(E) \cup \Delta_B$ is a basis of $\mathbb{Q}(\mathcal{B})$. Equivalently,

$$\mathbb{Q}(\mathcal{B}) = \mathrm{Mod}\big(\tau(E) \cup \Delta_B\big)$$

Definition

A sentential logic is $S = \langle \mathfrak{F}, \vdash \rangle$, where F is the absolutely free algebra generated by countably many variables, and $\vdash \subseteq \mathcal{P}(F) \times F$ satisfies

- S1 $\Gamma \vdash \phi$ if $\phi \in \Gamma$.
- S2 If $\Gamma \vdash \phi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \phi$.
- S3 If $\Gamma \vdash \phi$ and $\Delta \vdash \Gamma$, then $\Delta \vdash \phi$.
- S4 If $\Gamma \vdash \phi$ and $\sigma : \mathfrak{F} \to \mathfrak{F}$ is a homomorphism, then $\sigma[\Gamma] \vdash \sigma(\phi)$.

S is finitary if whenever $\Gamma \vdash \phi$ there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \phi$.

Definition

The sentential logic $S = \langle \mathfrak{F}, \vdash \rangle$ is equivalential if there is a set $E(x, y)$ of formulas in two variables x and y such that

- (E1) $\vdash E(\varphi, \varphi)$
- (E2) $E(\varphi, \psi) \vdash E(\psi, \varphi)$
- (E3) $E(\varphi, \psi), E(\psi, \vartheta) \vdash E(\varphi, \vartheta)$
- (E4) $E(\varphi_1, \psi_1), \ldots, E(\varphi_n, \psi_n) \vdash E(f(\varphi_1, \ldots, \varphi_n), f(\psi_1, \ldots, \psi_n))$ for any connective f.
- (E5) $\varphi, E(\varphi, \psi) \vdash \psi$

S is algebraizable if in addition there are ε, δ such that

- (A) $E(\varepsilon(\varphi), \delta(\varphi)) \dashv\vdash \varphi$.

Definition

A logical matrix is a pair $\langle \mathcal{A}, D \rangle$ where $\mathcal{A}$ is an algebra, and $D \subseteq A$ is a subset of designated elements. The matrix $\langle \mathcal{A}, D \rangle$ is a model for the sentential logic $S = \langle \mathfrak{F}, \vdash \rangle$ if

$$\Gamma \vdash \varphi \quad \Rightarrow \quad \forall h \in \mathrm{Hom}(\mathfrak{F}, \mathcal{A}) \ \big(h[\Gamma] \subseteq D \Rightarrow h(\varphi) \in D\big)$$

A class M of matrices is complete for S (provides a matrix semantics for S) if

$$\Gamma \vdash \varphi \quad \Leftrightarrow \quad \forall \langle \mathcal{A}, D \rangle \in M \ \ \forall h \in \mathrm{Hom}(\mathfrak{F}, \mathcal{A}) \ \big(h[\Gamma] \subseteq D \Rightarrow h(\varphi) \in D\big)$$

Proposition

For any non-empty class K of semi-Heyting algebras $\langle \mathfrak{F}, \vdash_K \rangle$ is an algebraizable logic with

$$E(x,y) = \{(x \to y) \wedge (y \to x)\}, \quad \varepsilon(\varphi) = \varphi, \quad \delta(\varphi) = 1\,.$$

Theorem

For any non-empty class K of semi-Heyting algebras the largest class of algebras for which $\langle \mathfrak{F}, \vdash_K \rangle$ is complete is $\mathbb{SP}(K)$.

In particular, for a finite semi-Heyting algebra $\mathcal{A}$, the quasi-variety $\mathbb{Q}(\mathcal{A})$ provides the equivalent algebraic semantics for $\vdash_{\mathcal{A}}$.

Definition

Let $A = \langle \mathfrak{F}, \vdash_A \rangle$ and $B = \langle \mathfrak{F}, \vdash_B \rangle$ be two logics having the same formulas $\mathfrak{F}$. A and B are deductively equivalent if there exist two translations σ_1 from A to B and σ_2 from B to A such that for all $\Gamma \cup \{\phi\} \subseteq \mathfrak{F}$,

(i) $\Gamma \vdash_A \phi$ iff $\sigma_1(\Gamma) \vdash_B \sigma_1(\phi)$, where $\sigma_1(\Gamma) := \{\sigma_1(\psi) : \psi \in \Gamma\}$;

(ii) $\sigma_1(\sigma_2(\phi)) \dashv\vdash_B \phi$.

Figure: Connections between the logics and algebras

$$
\begin{tikzcd}
\mathcal{L}_j \arrow[rr, leftrightarrow, "\text{term-equivalence}"] \arrow[d, leftrightarrow, "\text{algebraization}"'] & & \mathcal{L}_i \arrow[d, leftrightarrow, "\text{algebraization}"] \\
\mathcal{SL}_j \arrow[rr, leftrightarrow, "\text{deductive-equivalence}"] & & \mathcal{SL}_i
\end{tikzcd}
$$

Throughout different kinds of implication symbols are used:

- $\rightarrow$ means general semi-Heyting implication.
- $\Rightarrow$ means Heyting implication and is used when describing $\mathcal{L}_1$.
- $\stackrel{i}{\rightarrow}$ with $i \in \{2, ..., 10\}$ means implication of the corresponding algebra $\mathcal{L}_i$, for example $\stackrel{6}{\rightarrow}$ is the implication of $\mathcal{L}_6$.

Lemma

Algebras $\mathcal{L}_1 = \langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ *and* $\mathcal{L}_i = \langle A, \vee, \wedge, \xrightarrow{i}, 0, 1\rangle$ *for* $i \in \{2, 3, 4, 9, 10\}$ *are term-equivalent by the maps* τ *and* σ_i*, where:*

1. $\vee$, $\wedge$, *0 and 1 are mapped to themselves.*
2. $\tau : (x \Rightarrow y) \mapsto (x \xrightarrow{i} (x \wedge y)$.
3. $\sigma_2 : (x \xrightarrow{2} y) \mapsto (y \vee ((x \Rightarrow y) \wedge (y \Rightarrow x)))$.
4. $\sigma_3 : (x \xrightarrow{3} y) \mapsto (y \vee (x \Rightarrow 0))$.
5. $\sigma_4 : (x \xrightarrow{4} y) \mapsto (((x \Rightarrow 0) \wedge y) \vee ((x \Rightarrow y) \wedge (y \Rightarrow x)))$.
6. $\sigma_9 : (x \xrightarrow{9} y) \mapsto ((x \Rightarrow y) \wedge ((x \Rightarrow 0) \Rightarrow (y \Rightarrow 0)))$.
7. $\sigma_{10} : (x \xrightarrow{10} y) \mapsto ((x \Rightarrow y) \wedge (y \Rightarrow x))$.

Theorem

$\mathbb{V}(\mathcal{H}_3) = \mathbb{Q}(\mathcal{H}_3)$.

Lemma

The logic $\mathcal{SL}_1 = \langle \mathfrak{F}, \vdash_1 \rangle$, axiomatized by:

1. $\phi \Rightarrow (\psi \Rightarrow \phi)$.
2. $(\phi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \chi))$.
3. $\phi \wedge \psi \Rightarrow \phi$.
4. $\phi \wedge \psi \Rightarrow \psi$.
5. $\phi \Rightarrow \phi \vee \psi$.
6. $\psi \Rightarrow \psi \vee \phi$.
7. $(\phi \Rightarrow \chi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\phi \vee \psi \Rightarrow \chi))$.
8. $\perp \Rightarrow \phi$.

and the rule Modus Ponens, is algebraized by $\mathcal{L}_1 = \mathcal{H}_3$.

Lemma

The logic $\mathcal{SL}_2 = \langle \mathfrak{F}, \vdash_2 \rangle$, axiomatized by:

1. $\phi \xrightarrow{2} (\phi \wedge [\psi \xrightarrow{2} (\psi \wedge \phi)])$.
2. $[\phi \xrightarrow{2} (\phi \wedge [\psi \xrightarrow{2} (\psi \wedge \chi)])] \xrightarrow{2} ([\phi \xrightarrow{2} (\phi \wedge [\psi \xrightarrow{2} (\psi \wedge \chi)])] \wedge [(\phi \xrightarrow{2} (\phi \wedge \psi)) \xrightarrow{2} [(\phi \xrightarrow{2} (\phi \wedge \psi)) \wedge (\phi \xrightarrow{2} (\phi \wedge \chi))]])$.
3. $\phi \wedge \psi \xrightarrow{2} (\phi \wedge \psi \wedge \phi)$.
4. $\phi \wedge \psi \xrightarrow{2} (\phi \wedge \psi \wedge \psi)$.
5. $\phi \xrightarrow{2} (\phi \wedge (\phi \vee \psi))$.
6. $\psi \xrightarrow{2} (\psi \wedge (\psi \vee \phi))$.
7. $(\phi \xrightarrow{2} (\phi \wedge \chi)) \xrightarrow{2} [(\phi \xrightarrow{2} (\phi \wedge \psi)) \wedge [(\psi \xrightarrow{2} (\psi \wedge \chi)) \xrightarrow{2} ((\psi \xrightarrow{2} (\psi \wedge \chi)) \wedge [(\phi \vee \psi) \xrightarrow{2} ((\phi \vee \psi) \wedge \chi)])]]$.
8. $\bot \xrightarrow{2} (\bot \wedge \phi)$.
9. $\tau(\sigma(x \xrightarrow{2} y)) \leftrightarrow (x \xrightarrow{2} y)$.

and the rule Modus Ponens, is algebraized by $\mathcal{L}_2$.

Theorem

$\mathcal{L}_6$ and $\mathcal{L}_1$ are not term-equivalent.

Proof.

They have different subalgebras. $\square$

Theorem

The quasi-variety $\mathbb{Q}(\mathcal{L}_6)$ is a set of algebras satisfying:

1. *A set of equations defining the variety of semi-Heyting algebras.*
2. $x^* \vee (x \to y) \approx (x \vee y) \to y$,
3. $x \vee y \vee (x \to y) \approx x \vee [(x \to y) \to 1]$.
4. $0 \to 1 \approx 0$ *implies* $x \approx y$.
5. $0 \to 1 \approx 1$ *implies* $x \approx y$.

Lemma

The logic $\mathcal{SL}_6 = \langle \mathfrak{F}, \vdash_6 \rangle$, *axiomatized by:*

1. *Axioms of bounded lattice*
2. $[x \wedge (x \xrightarrow{6} y)] \leftrightarrow (x \wedge y)$.
3. $[x \wedge (y \xrightarrow{6} z)] \leftrightarrow (x \wedge [(x \wedge y) \xrightarrow{6} (x \wedge z)])$.
4. $x \xrightarrow{6} x$.
5. $x^* \vee x^{**}$.
6. $[x^* \vee (x \xrightarrow{6} y)] \leftrightarrow [(x \vee y) \xrightarrow{6} y]$.
7. $[x \vee y \vee (x \xrightarrow{6} y)] \leftrightarrow [x \vee [(x \xrightarrow{6} y) \xrightarrow{6} \top]]$.

and the rules:

- R1 *Modus Ponens.*
- R2 $(\bot \xrightarrow{6} \top) \leftrightarrow \bot \vdash_6 x \leftrightarrow y$.
- R3 $(\bot \xrightarrow{6} \top) \leftrightarrow \top \vdash_6 x \leftrightarrow y$.

is algebraized by $\mathcal{L}_6$.

Lemma

Algebras $\mathcal{L}_6 = \langle A, \vee, \wedge, \stackrel{6}{\to}, 0, 1\rangle$ *and* $\mathcal{L}_5 = \langle A, \vee, \wedge, \stackrel{5}{\to}, 0, 1\rangle$ *are term-equivalent by the maps* τ *and* σ*, such that:*

1. $\vee$, $\wedge$, *0 and 1 are mapped to themselves.*
2. $$\tau : (x \stackrel{6}{\to} y) \mapsto [(y \vee (y \stackrel{5}{\to} x)) \stackrel{5}{\to} (x \stackrel{5}{\to} y)].$$
3. $$\sigma : (x \stackrel{5}{\to} y) \mapsto [(x \stackrel{6}{\to} y) \wedge (x \vee y \vee (y \stackrel{6}{\to} x))].$$

Lemma

Algebras $\mathcal{L}_6 = \langle A, \vee, \wedge, \overset{6}{\to}, 0, 1\rangle$ *and* $\mathcal{L}_8 = \langle A, \vee, \wedge, \overset{8}{\to}, 0, 1\rangle$ *are term-equivalent by the maps* τ *and* σ*, such that:*

1. $\vee$, $\wedge$, *0 and 1 are mapped to themselves.*
2. $$\tau : (x \overset{6}{\to} y) \mapsto [[(0 \overset{8}{\to} 1) \vee ((0 \overset{8}{\to} y) \overset{8}{\to} (x \wedge y))] \overset{8}{\to} ((0 \overset{8}{\to} 1) \overset{8}{\to} (x \overset{8}{\to} y)$$
3. $$\sigma : (x \overset{8}{\to} y) \mapsto [(x \overset{6}{\to} y) \wedge ((x \wedge y) \vee ((x \overset{6}{\to} y) \overset{6}{\to} (0 \overset{6}{\to} y)))].$$

Lemma

Algebras $\mathcal{L}_6 = \langle A, \vee, \wedge, \xrightarrow{6}, 0, 1\rangle$ *and* $\mathcal{L}_7 = \langle A, \vee, \wedge, \xrightarrow{5}, 0, 1\rangle$ *are term-equivalent by the maps* τ *and* σ*, such that:*

1. $\vee$, $\wedge$, *0 and 1 are mapped to themselves.*
2. $$\tau : (x \xrightarrow{5} y) \mapsto [x \xrightarrow{7} (y \wedge (x \xrightarrow{7} y))].$$
3. $$\sigma : (x \xrightarrow{7} y) \mapsto [x \xrightarrow{8} (y \vee ((y \xrightarrow{8} x) \xrightarrow{8} 0))].$$

Theorem

The logics algebraized by $\mathcal{L}_i$ for $i \in \{5, 6, 7, 8\}$ are deductively equivalent.

Summary

- Among 10 3-element semi-Heyting algebras there are two non-term-equivalent varieties.
- Sankappanavar's definition of semi-Heyting algebras does not simplify to just intuitionistic logic.
- Open problem: For given set of n-element semi-Heyting chains, how many non-term-equivalent varieties can be generated by the set of these algebras?

Thank you for your attention!