Interpolation in bi-intuitionistic logics

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Interpolation in extensions

Craig and deductive Interpolation

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It is a classical logical problem. It can be seen as a geometric definability theorem: it concerns the (sub)-definability of intersections.

For these logics, interpolation is very closely related to amalgamation.

Definition (Amalgamation)

A triple $B_2 \hookleftarrow A \hookrightarrow B_1$ is called a V-formation. An algebra C is called an amalgam if the diagram commutes:

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A class **V** has (super)amalgamation if each V-formation (super)amalgamates.

Interpolation and amalgamation

In general for our purposes we have the following general equivalence:

Theorem ((Craig) Interpolation = (Super)Amalgamation, Maksimova)

A logic L has:

1. *Deductive interpolation if and only if* $\mathrm{Alg}(L)$ *has amalgamation;*
2. *Craig interpolation if and only if* $\mathrm{Alg}(L)$ *has superamalgamation.*

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Depends crucially on the congruence extension property/local deduction theorem, and Beth definability properties:

1. **CRing** has (uniform!) interpolation but fails amalgamation;
2. **DLat** has a suitable Craig interpolation property, but fails superamalgamation.

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2. **(Model theory)**: study Fraissé classes and generic models in subvarieties;
3. **(Universal algebra)**: study strong epimorphism surjectivity properties;
4. **(Categorical logic)**: classify coreflective subcategories of $\mathbf{V}$, regularity of subcategories, regularity of monomorphisms.

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On the other hand:

Theorem (Anti-classification for GL)

There exist continuum many extensions of GL *with interpolation.*

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The language: $\mathcal{L}_{\text{biIPC}} = (\wedge, \vee, \rightarrow, \dot{-}, 0, 1)$; algebras are bi-Heyting algebras, lattices $(H, \leq)$ having, for each $a, b \in H$, elements such that for each c:

$$c \wedge a \leq b \iff c \leq a \rightarrow b$$
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Extensions are closed super b.i. logics. A particular example is the biGD logic of Martins and Moraschini:

$$\mathrm{biGD} = \mathrm{biIPC} \oplus (p \to q) \vee (q \to p).$$

Its algebras are *bi-Gödel algebras*. It appears naturally in several contexts in the study of trees.

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Today:

1. A classification of super b.i. logics with (Craig/deductive) interpolation does not seem plausible.
2. We give a full classification for extensions of biGD.

Interpolation in bi-intuitionistic logics

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\text{bi-Heyting algebras} &\equiv \text{Posets with order-topology}\\
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$$\begin{tikzcd} & Z \arrow[dl, dashed, two heads] \arrow[dr, dashed, two heads] & \\ Y_1 \arrow[dr, two heads] & & Y_2 \arrow[dl, two heads] \\ & X & \end{tikzcd}$$

Co-amalgamation.

A simple example of this:

Theorem

The logic biIPC *has the Craig interpolation property.*

Proof.
Given $f\colon Y_1 \to X$ and $g\colon Y_2 \to X$ surjective maps, we take:

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The projections will be bi-p-morphisms, and surjective. Then we use the embedding of $\mathrm{ClopUp}(Y_1 \times_X Y_2) \hookrightarrow \mathrm{Up}(Y_1 \times_X Y_2)$, which preserves all existing Heyting and co-Heyting structure. □

Too many extensions for classification

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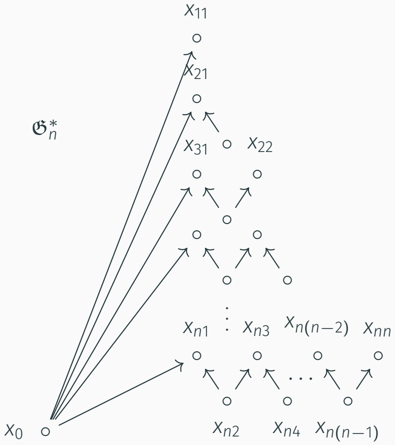

Perhaps not too many with CIP

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Question: is there a reasonable combinatorial criterion determining which extensions of biIPC have the CIP?

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Theorem (R.N.A, Qian Chen)

There are exactly six extensions of biGD *with DIP:*

$$\mathrm{biGD}, \mathrm{biLC}, \mathrm{biGDBD}_2, \mathrm{biF}_2, \mathrm{biLC}_2, \mathrm{CPC}.$$

All except for biGDBD_2 *and* biF_2 *have the CIP.*

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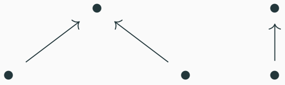
biLC – logic of *linear frames*; biLC_2 – chains of height at most 2;

biGDBD_2 – logic of co-forks: depth at most 2; biF_2 – logic of the 2-fork.



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For biGD we employ Carai's (2026) representation of free Gödel algebras:

Theorem (Carai, 2026)

If X, Y are Esakia spaces dual to Gödel algebras, then the categorical product of them in this category is given by:

$$\mathsf{CC}_{\mathrm{up}}(X \times Y) = \{C \subseteq X \times Y : \ C \textit{ is a closed chain}, \pi_i[C] \textit{ is an upset } \}.$$

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We describe categorical pullbacks of Gödel algebras, and use a similar proof strategy as for biIPC, made more difficult by needing to verify linearity and bi-p-morphism conditions.

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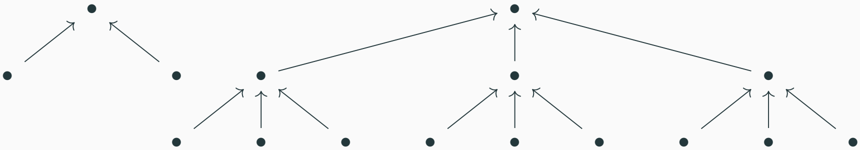
Similar to Maksimova's Jaskowski-like trick: we use full n-ary trees.



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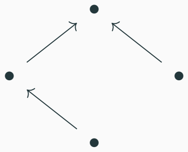
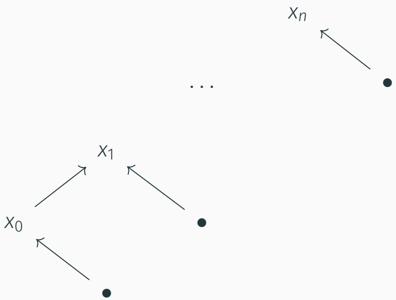
- If L has depth 1, then it is CPC and we are done.
- If it has depth 2, either it is characterized by C_2 or by F_2, or it must contain all forks – it must be biGDBD_2.
- If it has depth > 2 and width 1, then it contains a 3-element chain, it must contain all chains – it must be biLC;
- Otherwise, it has depth > 2 and width > 1: by amalgamation, it contains a *basic 2-comb* Comb_2, and by amalgamation again, all combs Comb_n.



Logic of finite combs

n-combs are the structures which generalize the Comb_2-structure:



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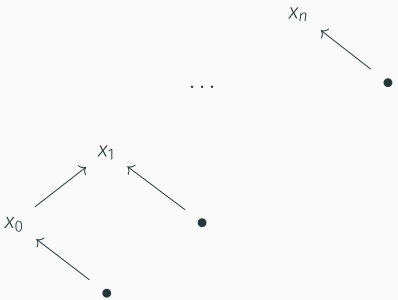

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The logic LFC, *generated by all finite combs is the unique pre-locally tabular logic.*

I.e., LFC is *not locally tabular but all of its extensions are.*

Proof of the classification

4. Show that if L contains a 2-comb (or a 3-element chain) and a 3-fork, and has the DIP, then L contains all the frames from $\mathfrak{B}$.

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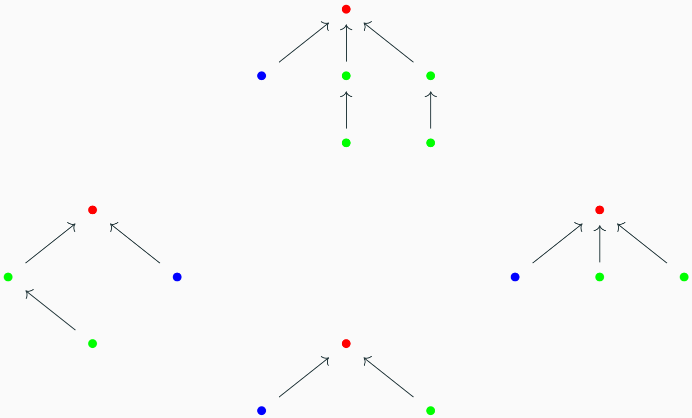

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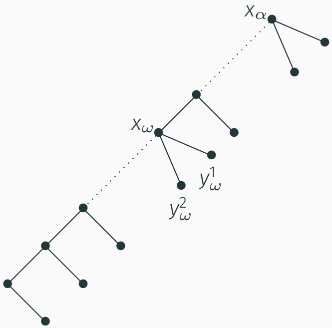
We need unreasonably large Esakia spaces!

Proof of the classification

Given an ordinal α, the doubled α-comb:



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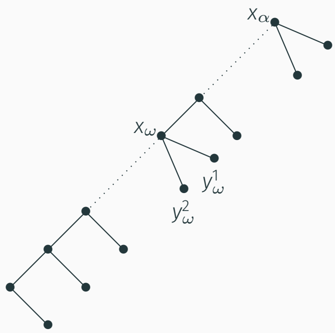

At limit stages, there are always two copies on the teeth. We denote these combs by $\mathfrak{Cb}_\alpha$.

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The diagram we consider is then:

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Key idea: cofinality argument over ω_1 ensures that we can create the 3-fork.

Final remarks

Future work

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2. What about uniform interpolation? This fails (in the global sense) for biIPC. It might work for biGD by working over Gödel algebras.

Thank you!

Questions?