# Semiring-valued logics with twist-structure semantics

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- Address some known issues with negation and implication in semiring-valued logics
- Apply an FDE-style semantics to deal with these issues
- Show basic properties of the resulting logics and their variations

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# Semirings

Recall: A semiring is an algebraic structure $\boldsymbol{S} = (S, +, \cdot, 0, 1)$ s.t.

1. $(S, +, 0)$ is a commutative monoid
2. $(S, \cdot, 1)$ is a monoid
3. $x(y+z) = xy + xz, \; (y+z)x = yx + zx$
4. $0x = x0 = 0$

The class SR of semirings includes, eg:

- Rings, fields $(\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}_n, \dots)$, other arithmetic semirings $(\mathbb{N}, \mathbb{R}_0^+, \dots)$
- Bounded distributive lattices, bounded residuated lattices (BA, HA, $\mathrm{FL}_w$, MV, chains, ...)
- Semiring-valued matrices, polynomials, formal power series, ...

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# Why semirings in logic?

Semiring values = annotations of program runs / process flows
(resources spent, transitions taken, ... )

The properties of semiring operations reflect the procedural motivation:

- $+$ = aggregation of alternatives (choice, parallel branches)
- $\cdot$ = accumulation along a process (serial composition)
- $0$ = failure/falsity (unaffordable cost, impossible transition)
- $1$ = 'no-op' (no cost, no annotation, no state-change)

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# Semiring-valued semantics

An established program of semiring-valued logic: Semiring semantics
(Grädel, Tannen, and others)

First-order semiring semantics:

- $[\![P\vec{x}]\!] : D^n \to S$, $[\![\vee]\!] = +^S$, $[\![\wedge]\!] = \cdot^S$, $[\![\bot]\!] = 0^S$
- $0 \neq a =$ "a nuanced/annotated notion of truth" (Grädel & Tannen)

Acknowledged issues with $\neg, \to$:

- No satisfactory (general) definition (semirings $\approx$ positive logic only)
- Treated only by restrictions on semirings or on placement
  (eg, $\to$ only in suitably ordered SR, $\neg$ only in literals, ...)

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# Semiring-based dual-valuation algebras

Proposal: Separate evaluation of positive and negative evidence

$\Rightarrow$ Evaluate formulas by *two* semiring values, $[\![\varphi]\!] = (t_\varphi, f_\varphi) \in S^2$

$t_\varphi$ = positive support (verification, confirmation, derivations of $\varphi$)

$f_\varphi$ = negative support (refutation, counterevidence, derivations against $\varphi$)

Dual-valuation algebra = semiring twist-structure $\boldsymbol{S}^{\bowtie}$:

$\mathrm{DVA}(\boldsymbol{S}) = (S \times S, \vee, \wedge, \neg)$ over a semiring $\boldsymbol{S} = (S, +, \cdot, 0, 1)$:

$$(t_x, f_x) \vee (t_y, f_y) = (t_x + t_y,\ f_x \cdot f_y) \qquad \text{(choice / both)}$$

$$(t_x, f_x) \wedge (t_y, f_y) = (t_x \cdot t_y,\ f_x + f_y) \qquad \text{(both / choice)}$$

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- Procedures for verification and refutation often differ

  (eg, in logic: search for a proof / counterexample)

  $\Longrightarrow$ truth-functional negation on $S$ rarely adequate

- Dual valuations have already been suggested (but not developed) for handling negation in semiring semantics by Grädel & Tannen (2025)

  Their *dual-indeterminate provenance semantics* essentially induces DVAs

  (by independently evaluating positive and negative literals)

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## Basic properties of semiring-based DVAs

The class DVA(SR) includes some well-known algebras (modulo signature):

- FOUR = DVA($\mathbf{2}$), NINE = DVA($\mathbf{3}$), ...
- Bounded product bilattices, square MV-algebras, ...

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**Valid identities:**

- $\neg\neg x \approx x$ — involution
- $(x \vee y) \vee z \approx x \vee (y \vee z)$, $(x \wedge y) \wedge z \approx x \wedge (y \wedge z)$ — associativity
- $\neg(x \vee y) \approx \neg x \wedge \neg y$, $\neg(x \wedge y) \approx \neg x \vee \neg y$ — De Morgan

$\Rightarrow$ Negation normal form: $\tau \approx \tau^{\mathrm{NNF}}$ ($\neg$ only in literals)

**Invalid identities:**

- Commutativity of $\vee, \wedge$ (valid iff over commutative semirings)
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# Orthogonal DVAs

DVAs admit paraconsistency: both channels can be non-failing (ie, $\neq 0$)

$\Rightarrow$ Orthogonality constraint:

$$t_\varphi = 0 \quad \text{or} \quad f_\varphi = 0$$

The axis subalgebra of DVA = orthogonal DVA (oDVA)

Example: K3 = oDVA($\mathbf{2}$)

Not many interesting additional identities

(though significant differences wrt consequence relations)

Example: $\text{oDVA(SR)} \models (x \vee \neg x) \vee \neg(x \vee \neg x) \approx x \vee \neg x \not\models \text{DVA(SR)}$

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# Consequence relations over DVAs

There is not much structure in semiring-based DVAs,
but still some meaningful consequence relations are definable

Example: preservation of confirmability

Recall: $0 =$ fail, $t_x =$ confirmation value $\implies$ "$x$ confirmable" iff $t_x \neq 0$

Confirmability-preserving consequence relation: $\Gamma \models^{t\neq 0} \varphi$ iff:

$$\forall \boldsymbol{S} \in \mathrm{SR}, \forall \, \mathrm{DVA}(\boldsymbol{S})\text{-val.:} \quad (\forall \gamma \in \Gamma)(t_\gamma \neq 0) \implies t_\varphi \neq 0$$

$=$ Dual-valuation logic $\mathrm{DVL(SR)}^{t\neq 0}$

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# Confirmability-preserving entailment over DVAs

Basic properties of $\models^{t\neq 0}$:

- Substitution-invariant Tarski (matrix) consequence relation
- $\mathrm{DVL(SR)}^{t\neq 0} \subsetneq \mathrm{FDE}$ (FOUR is a model)
  $\Longrightarrow$ no tautologies $\Longrightarrow$ not protoalgebraic
- Does not have *almost parametrically equationally definable truth*
  (FOUR has two filters $\{(1,0),(1,1)\}$, $\{(1,0),(0,0)\}$ that are models)
- Not self-extensional (logical equivalence $\not\Longrightarrow$ intersubstitutivity)
- Has the variable-sharing property
- Non-commutative DNF: $\varphi \equiv^{t\neq 0} \bigvee_i \bigwedge_j \lambda_{ij}$ (literals)

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## Valid rules:

- DVA-identities: De Morgan, associativity, double negation
- $\wedge$-Elimination: $\varphi \wedge \psi \models \varphi$, $\varphi \wedge \psi \models \psi$
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Valuations restricted to the axes (oDVAs = 'consistency'):
orthogonal dual-valuation logics (oDVLs)

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- $\mathrm{oDVL(SR)}^{t\neq 0} \subsetneq \mathrm{K3}$ (K3 is a model)
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- Preservation of irrefutability (ie, of $f = 0$)
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Dual-valuation logics over special classes of semirings

(commutative, idempotent, simple, positive, ...)

$\Longrightarrow$ Additional valid rules:

- $\wedge, \vee$ commutative in DVA(CSR), idempotent in DVA(IdSR), ...
- Adjunction valid for $\models^{t \neq 0}$ over *entire* semirings (ie, no 0-divisors), etc

Expansions of the language

- Truth constants $(\mathbf{t}, \mathbf{f}, \mathbf{b}, \mathbf{n})$ $\Longrightarrow$ tautologies, expressive power, ...
- Indicators (eg, $\nabla$ indicating $t \neq 0$, consistency operator $\circ$, etc)
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# Symmetric dual-valuation algebras and logics

Note the fixed evaluation order in DVAs: $t_{\varphi\wedge\psi} = t_\varphi \cdot t_\psi, \quad f_{\varphi\vee\psi} = f_\varphi \cdot f_\psi$

Sometimes OK (Lambek calculus, weighted automata, dynamic semantics)

Sometimes *not OK* $\implies$ symmetrization/choice of evaluation order:

(a) Pairwise symmetrization (loses associativity):

$$t_{\varphi\wedge\psi} = t_\varphi t_\psi + t_\psi t_\varphi\,, \qquad f_{\varphi\vee\psi} = f_\varphi f_\psi + f_\psi f_\varphi$$

(b) Full symmetrization (complex semantics):

$$t_{\bigwedge_{i\le n} A_i} = \sum_{\pi\in S_n} \prod_{i\le n} t_{A_{\pi(i)}}\,, \qquad f_{\bigvee_{i\le n} A_i} = \sum_{\pi\in S_n} \prod_{i\le n} f_{A_{\pi(i)}}$$

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# Implication

(a) Material implications: $x \supset y \equiv \neg x \vee y, \quad y \subset x \equiv y \vee \neg x$

Valid identities:

- Contraposition: $x \supset y \approx \neg x \subset \neg y$
- Currying: $x \supset (y \supset z) \approx x \wedge y \supset z$

Bad inferential properties (eg, for $\models^{t \neq 0}$):

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# Deductive implication

(c) Deductive implication $\to$, which internalizes $\models$ via DDT (ie, satisfies $\Gamma \models \varphi \to \psi \iff \Gamma, \varphi \models \psi$)

Can be added to every DVA for every non-trivial consequence $\models^{\mathcal{M}}$

Example: For the confirmability consequence $\models^{t \neq 0}$, define $\to$, eg, as

$$(t_\varphi, f_\varphi) \to (t_\psi, f_\psi) =_{\mathrm{df}} \begin{cases} (1,0) & \text{if } t_\varphi = 0 \text{ or } t_\psi \neq 0 \\ (0,1) & \text{if } t_\varphi \neq 0 \text{ and } t_\psi = 0 \end{cases}$$

Inferentially good:

- Validates modus ponens, hypothetical syllogism, ...
- Captures inference under $\models^{\mathcal{M}}$
- Is just two-valued, but $\{0,1\}$-*valued* $\neq$ *Boolean* in general over SR

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