Abelian Framed Bicategories

A Category of Modules for Directed Topology

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- 0-cells are rings,
- vertical 1-cells $A \to B$ are ring morphisms
- horizontal 1-cells $M : A \nrightarrow B$ are A-B-bimodules,
- 2-cells $\begin{tikzcd} A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "\shortmid"{marking}] & D \end{tikzcd}$ are f, g-bilinear maps $\alpha : M \to N$, i.e., morphisms of abelian group satisfying $\alpha(a \cdot m \cdot b) = f(a) \cdot \alpha(m) \cdot g(b)$.

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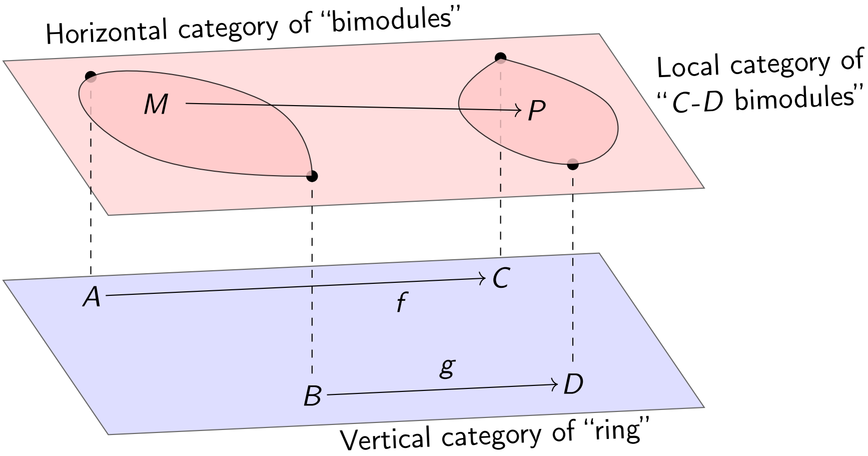
Definition (Ehresmann, "Catégories structurées")

A **double category** is an internal category in the category Cat of small categories, i.e., the data of two small categories $\mathbb{D}_0$ and $\mathbb{D}_1$ and functors:

$$U : \mathbb{D}_0 \to \mathbb{D}_1 \qquad L, R : \mathbb{D}_1 \to \mathbb{D}_0 \qquad \odot : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$$

satisfying the usual category laws (suitably weakened).

The double category of modules over rings



Restriction & extension of scalars, abelian fibers

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- A bimodule $M : A \nrightarrow B$ can be **restricted** along ring morphisms $f : A' \to A$ and $g : B' \to B$: the restriction $f^*Mg^* : A' \nrightarrow B'$ is M with actions $a \cdot_{A'} M = f(a) \cdot_A M$ and $M \cdot_{B'} b = M \cdot_B g(b)$.

$$\begin{tikzcd} A' \arrow[r, "f^*Mg^*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & B' \arrow[d, "g"] \\ A \arrow[r, "M"', "\shortmid"{marking}] & B \end{tikzcd}$$

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- Similarly, there is an **extension** $f_! M g_! : A' \nrightarrow B'$ of $M : A \nrightarrow B$ along ring morphisms $f : A \to A'$ and $g : B \to B'$.

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- Similarly, there is an **extension** $f_! M g_! : A' \nrightarrow B'$ of $M : A \nrightarrow B$ along ring morphisms $f : A \to A'$ and $g : B \to B'$.
- Over a fixed pair of rings A, B, the "local" category of A-B-bimodules is abelian, but the "total" category of bimodules over varying rings is *not*: $0 : \mathbb{Z} \nrightarrow \mathbb{Z}$ is initial and $0 : 0 \nrightarrow 0$ is terminal.

Framed bicategories

Definition (Shulman, "Framed bicategories and monoidal fibrations")

A **framed bicategory** $\mathbb{D}$ is a double category such that the functor $(L, R) : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is a bifibration,

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$$\begin{tikzcd} A \arrow[r, "f^*Mg^*", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "cart"] & C \arrow[d, "g"] \\ B \arrow[r, "M"', "\shortmid"{marking}] & D \end{tikzcd}$$

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i.e., any 2-cell $\begin{tikzcd} A' \arrow[r, "N", "|" marking] \arrow[d, "fh"'] \arrow[dr, phantom, "\Downarrow"] & C' \arrow[d, "gk"] \\ B \arrow[r, "M"', "|" marking] & D \end{tikzcd}$ factors uniquely as $\begin{tikzcd} A' \arrow[r, "N"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow"] & C' \arrow[d, "k"] \\ A \arrow[r, phantom, "f^*Mg^*"] \arrow[d, "f"'] \arrow[dr, phantom, "cart"] & C \arrow[d, "g"] \\ B \arrow[r, "M"', "|" marking] & D \end{tikzcd}$.

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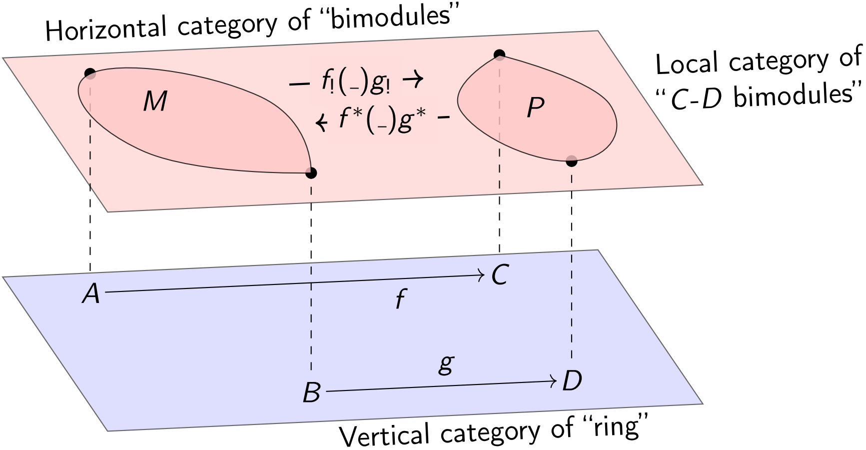
There are **cocartesian** 2-cells as well. . .

Framed bicategories



Abelian framed bicategories

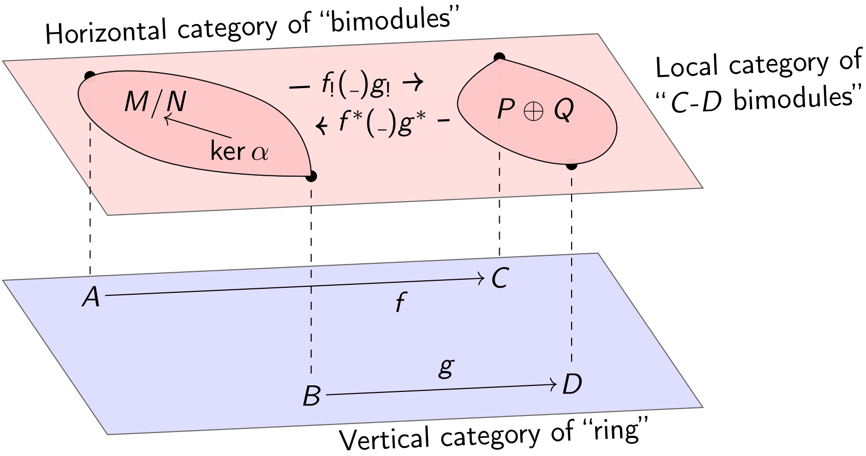
Definition (Albert, Dubut, Goubault, "Homological algebra in abelian framed bicategories")

An **abelian framed bicategory** is a framed bicategory $\mathbb{D}$ that is locally abelian, i.e., all local categories $\mathcal{D}(A, B)$ are abelian, and such that horizontal composition yields additive functors:

$$\begin{aligned} \mathcal{D}(A, B) \times \mathcal{D}(B, C) &\to \mathcal{D}(A, C) \\ (M, N) &\mapsto M \odot N \end{aligned}$$

for all objects $A, B, C \in \mathbb{D}_0$.

Abelian framed bicategories



Homology in Abelian framed bicategories[1]

A **chain complex** in $\mathbb{A}$ is a chain complex in a local abelian category of $\mathbb{A}$.

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A **chain complex** in $\mathbb{A}$ is a chain complex in a local abelian category of $\mathbb{A}$.
A **morphism** $(\alpha_i)_{i\geq 0}$ from $(M_i)_{i\geq 0}$ in $\mathcal{A}(A,B)$ to $(N_i)_{i\geq 0}$ in $\mathcal{A}(C,D)$ is:

- a pair of arrows $f : A \to C$ and $g : B \to D$,
- 2-cells $\alpha_i : M_i \overset{g}{\underset{f}{\Longrightarrow}} N_i$

such that the following commutes for all $i \geq 0$:

$$\begin{tikzcd}[row sep=huge, column sep=huge]
A \arrow[r, bend left, "M_i"{name=Mi}] \arrow[r, bend right, "M_{i+1}"'{name=Mi1}] \arrow[r, phantom, "\partial_{i+1}\Uparrow"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r, bend left, "N_i"{name=Ni}] \arrow[r, bend right, "N_{i+1}"'{name=Ni1}] \arrow[r, phantom, "\Uparrow\partial_{i+1}"] & D
\arrow[from=Mi1, to=Ni1, Rightarrow, "\alpha_{i+1}"'] \arrow[from=Mi, to=Ni, Rightarrow, "\alpha_i"]
\end{tikzcd}$$

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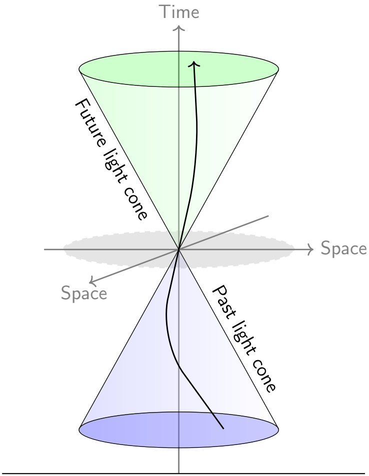
2. It has nice homological properties: long exact sequences for relative pairs, Künneth, Mayer-Vietoris, and embedding theorems, etc.
3. It spontaneously appears in **directed topology**, where the above properties provide non-trivial results.

Directed topology[2][3]

"Directed spaces have privileged directions, therein directed paths need not be reversible"

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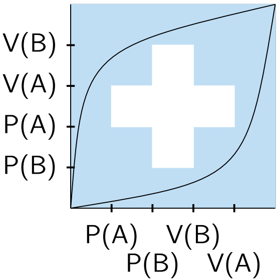
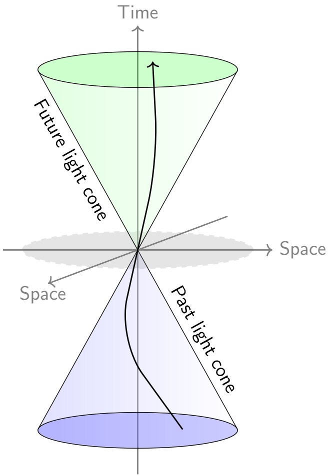


[2]Grandis, *Directed Algebraic Topology: Models of Non-Reversible Worlds*.
[3]Fajstrup et al., *Directed algebraic topology and concurrency*.

Directed spaces

Definition

A **directed space** is a topological space X along with a subset of paths $dX \subset X^{[0,1]}$ such that:



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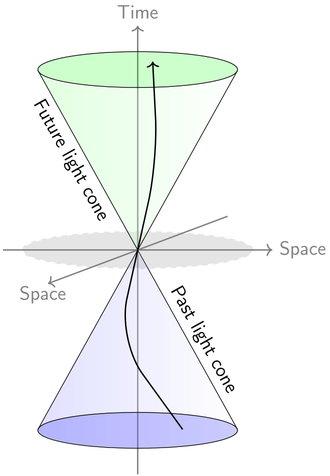

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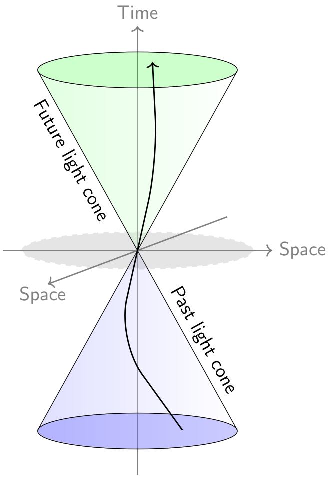

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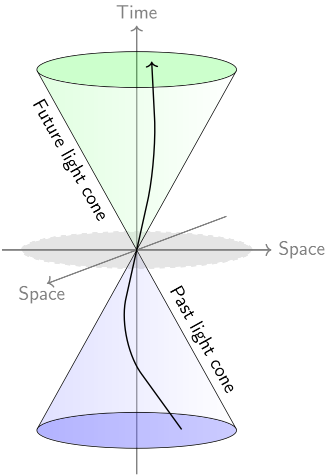
A **directed space** is a topological space X along with a subset of paths $dX \subset X^{[0,1]}$ such that:

- dX contains constant paths
- dX is stable under concatenation
- dX is stable under reparameterization by increasing maps $[0,1] \to [0,1]$



Another application : control systems[4]

The solutions of a differential inclusion $x'(t) \in F(t, x(t))$ in $\mathbb{R}^n$ induce a directed structure on $\mathbb{R}^n$.

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Theorem

Directed topological complexity *quantifies the minimal number of "discontinuities" in a section of*

$$X^I \to X \times X$$
$$p \mapsto (p(0), (p(1))$$

i.e. a solution to the path finding problem.

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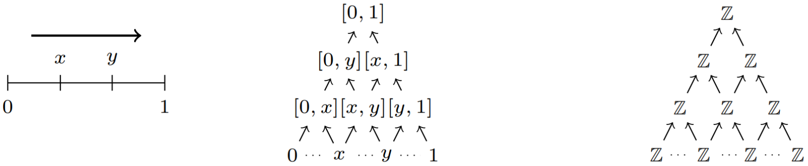
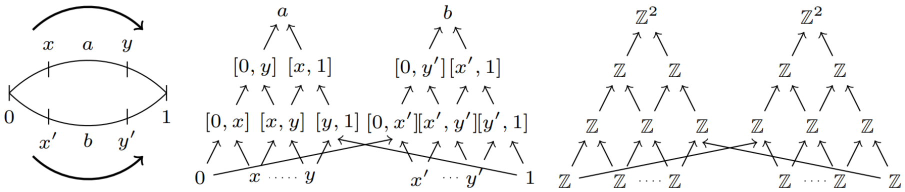
Natural Homology[5][6]

A homological invariant for directed topology

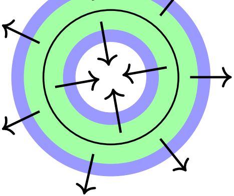




[5] Dubut, Goubault, and Goubault-Larrecq, "Natural homology".
[6] Goubault, "Directed homology and persistence modules".

A long exact sequence for relative pairs

An example property of homology in abelian framed bicategories



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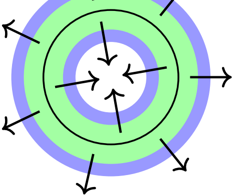

- Consider $Y \subseteq X \in \mathrm{dTop}$ and a functor $M : \mathrm{dTop} \to \mathrm{Ch}(\mathbb{A})$ (e.g. take ${}_R\mathbb{M}\mathrm{od}_R$ the abelian framed bicategory of bimodules over R-algebras).

➔ $H_*(X) \in \mathcal{A}(X)$ and $H_*(Y) \in \mathcal{A}(Y)$ are over different coefficients...

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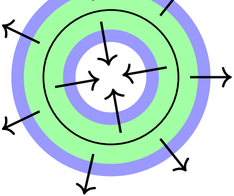

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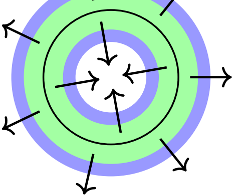

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Theorem

For all ***relative pairs*** (X, Y)*, there is a long exact sequence for relative homology:* $\cdots \to H_{i+1}(X, Y) \to {}^X H_i(Y) \to H_i(X) \to H_i(X, Y) \to \cdots$.

Relative pairs

For $M : \mathcal{C} \to \mathrm{Ch}(\mathbb{A})$

Definition

$f : Y \to X \in \mathcal{C}$ is a relative pair when

- extension of scalars ${}^{X}(\_)$ is exact,
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Example

An inclusion of directed paths $Y \hookrightarrow X$ is a relative pair iff a directed path in X can enter and exit Y at most once.

An embedding theorem

Theorem (Gabriel ("Unzerlegbare darstellungen I"))

Let $\mathcal{A}$ be a cocomplete abelian category. If $\mathcal{A}$ has a compact projective generator P, $\mathcal{A}$ is equivalent to ${}_{\mathrm{End}(P)}\mathrm{Mod}$.

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Theorem

If $\mathbb{A}$ is module-like, there is a framed lax functor $F : \mathbb{A} \to {}_R\mathrm{Mod}_R$ into a concrete abelian framed bicategory of modules over R-algebras, with $R = \mathrm{End}(U_I)$, such that F_1 is fully faithful and locally an equivalence of categories.

Perspectives

- Going from a Gabriel to a Freyd–Mitchell embedding theorem.
- Directed cohomology and lower bounds for directed topological complexity.
- Homotopy reconstruction theorems à la Cohen-Jones-Segal for directed spaces.

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