Cut-free Deductive Systems for Affine and Intuitionistic Continuous Logic

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Table of contents

Motivations

1. Lay the ground work for an analysis of metric structures internal to Grothendieck toposes.

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    1. Locales are internalisable into toposes.
    2. Internal notions of norms of Banach spaces or metric for metric spaces are valued into the semi-continuous real numbers.
    3. When the topos is localic, the externalisation of these real numbers is $USC(\mathscr{L}, \mathbb{R})$, where $\mathscr{L}$ is the locale of the topos.
2. Provide a general sequent-style calculus with a metric (to be linked with metric spaces of λ-terms).

AC-algebras

X a topological space.

An upper semi-continuous function from X to $[0,1]$ is a function $f\colon X \to [0,1]$ such that, for all $q \in [0,1]$, $f^{-1}([0,q))$ is open.

For a function $f\colon X \to [0,1]$, to be upper semi-continuous can be expressed by the property $f = \bigwedge_{q \in [0,1]} \underline{q} \dot{+} \mathbf{0}_{f(q)}$.

Dually, the set of upper semi-continuous functions can be defined as the set of (directed) upper bound preserving functions from $[0,1]$ to the topology of X.

AC-algebras

$\mathscr{L}$ a c.r.c.l. (commutative quantale).

Definition

An upper semi-continuous function from $\mathscr{L}$ to $[0,1]$ is an upper-bound preserving function $f : [0,1] \to \mathscr{L}$.

For $q \in [0,1]$, $f(q)$ is "the open $\{f < q\}$".

$f \leq g \Leftrightarrow \forall q \in [0,1],\ g(q) \leq f(q)$, $\quad \underline{1} = \max(USC(\mathscr{L}))$, $\quad \underline{0} = \min(USC(\mathscr{L}))$.

AC-algebras

$$(f \dot{+} g)(q) = \bigvee_{p \dot{+} r < q} f(p) \otimes g(r),\ f \dot{-} g \leq h \Leftrightarrow f \leq g \dot{+} h.$$

$$2f : q \to f\left(\frac{q}{2}\right),\ \frac{f}{2} \leq g \Leftrightarrow f \leq 2g,\ j(f) = 2\left(f \dot{-} \frac{1}{2}\right),\ j_*(f) = \frac{f}{2} \dot{+} \frac{1}{2},$$
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Definition

We call every $(USC(\mathscr{L}), \vee, \wedge, \dot{+}, \dot{-}, 2, \frac{\cdot}{2}, j_*, j, \alpha, \underline{0}, \underline{1})$, *for* $\mathscr{L}$ *a c.r.c.l., an Affine Continuous algebra, or AC-algebra.*

First result

Theorem

Let $\mathcal{L}_+ = \{\vee,\ \wedge,\ \dot{+},\ 2,\ \dot{\frac{}{2}},\ j_*,\ j,\ \alpha,\ \underline{0},\ \underline{1}\}$ *and* $(\varphi_i)_{1\leq i\leq n}$ *and* ψ *be terms in the language* $\mathcal{L}_+$ *and assume that, in each* φ_i*, any variable occurs at most once. Let* $\bar{\varphi}$ *be the* $\mathcal{L}_+$*-term obtained by replacing every occurrence of* $\vee$ *in* φ *by* $\dot{+}$*.*

If $[0,1] \models \bigwedge_{i=1}^{n} \varphi_i \leq \psi$*, then* $USC(\mathscr{L}) \models \bigwedge_{i=1}^{n} \bar{\varphi}_i \leq \bar{\psi}$*.*

Axiomatisation of AC-algebras

The theory T:

1. $(\vee, \wedge, \dot{+}, \dot{-}, \underline{0}, \underline{1})$ **is a bounded commutative residuated lattice structure.**

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$$\alpha(v) = \frac{v}{2} \vee j(v), \qquad \alpha^{-1}(v) = 2v \wedge j_*(v)$$

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5. **The algebra of values is** $[0,1]$ **up to infinitesimals.**

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The algebra of values is $[0,1]$ up to infinitesimals:

1. for all dyadic number $d \in [0,1]$ and $n \in \mathbb{N}$, $v \leq \underline{d} \dot{+} \alpha^n(v \dot{+} \underline{1-d})$
2. $\bigwedge_{\substack{d \in [0,1] \\ \textbf{dyadic}}} \underline{d} \dot{+} ?(v \dot{+} \underline{1-d}) \leq v \dot{+} \underline{\frac{1}{2^n}}$.

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Two non-algebraic properties:

1. Completeness : the partial order is complete
2. Archimedeanity : the lower bound of the family of $\underline{\frac{1}{2^n}}$ is $\underline{0}$

Main result

$a \preceq b \Leftrightarrow \forall n \in \mathbb{N}\ a \leq b \dot{+} \underline{\frac{1}{2^n}}$ defines a preorder on every model of T.

Theorem

For all model A of T, there exists a c.r.c.l. $\mathscr{L}$ such that the Archimedean quotient of the MacNeille completion of A is isomorphic to $USC(\mathscr{L})$.

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Idea of the proof.

1. The MacNeille completion of a model of T is a model of T.

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3. Every complete *Archi.* model of T_1 is a complete *Archi.* model of T_0, which is what remains of T_1 in a simpler language and every T_0-isomorphism between complete *Archi.* models of T_1 is a T_1-isomorphism.

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4. Every complete *Archi.* model of T_0 is "isomorphic" to a $USC(\mathscr{L})$. □

Consequences that can be found in the prepublication

1. A sequent-style calculus with the cut admissibility property such that the class of AC-algebras is complete for it.
2. Intuitionistic case: when $\mathscr{L}$ is a complete Heyting algebra, or equivalently when $2f = f \dot{+} f$. This is the starting point of intuitionistic continuous logic.
3. Involutive case: when the negation of $\mathscr{L}$, or equivalently the one of $USC(\mathscr{L})$ is involutive.
4. Boolean case: when $\mathscr{L}$ is a complete Boolean algebra. The logic obtained is classical continuous logic.

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