On the logics of finite Gödel Kripke models

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Joint work with R. Rodriguez

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A short motivational slide

- Semantics of modal Gödel logic can come from two directions: semilinear K-double models (Heyting modal structures), or Kripkpe models valued over Gödel algebras.

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- Semantics of modal Gödel logic can come from two directions: semilinear K-double models (Heyting modal structures), or Kripkpe models valued over Gödel algebras.
- It is known that the logics are the same. However, the second approach offers the definition of very intuitive ideas/frame conditions whose meaning under the first semantics might be more less intuitive (e.g., crisp accessibility).
- In most cases, the logics are not complete with respect to finite models in the second semantics (even if, at least the local logics, are known to be decidable). It has been an open problem to characterize the logics arising from finite models (with crisp or with valued accessibility).

Some definitions

Definition

An (standard valued) **Gödel Kripke model** $\mathfrak{M}$ is a structure $\langle W, R, e\rangle$ with W a non-empty set, $R\colon W^2 \to [0,1]$ and $e\colon W \times \mathcal{V} \to [0,1]$, with e being point-wise a Gödel homomorphism and furthermore,

$$e(v, \Diamond\varphi) := \bigvee_{w\in W} R(v,w) \wedge e(w,\varphi), \; e(v,\Box\varphi) := \bigwedge_{w\in W} R(v,w) \to e(w,\varphi)$$

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- MG is the logic of all Gödel Kripke models;
- KG is the logic of all crisp Gödel Kripke models;
- MG^ω, KG^ω are the logics of all corresponding finite models;
- $\mathcal{L}_\Box/\mathcal{L}_\Diamond$ refer to the corresponding mono-modal fragment of $\mathcal{L}$.

Some Definitions (witnessing conditions)

A model $\mathfrak{M}$ is **witnessed** if for every formula φ,

$$e(v, \Box\varphi) = R(v, w) \to e(w, \varphi) \text{ for some } w \in W,$$
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The local entailment over a class of witnessed models enjoys the FMP.

We say $\mathcal{L}$ **is witnessed** when it is complete with respect to a class of witnessed models.

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- **Can we get finite axiomatizations of the witnessed companions of the previous logics, and of $KG_{\diamond}$? (preliminar proof-theoretic studies in [4,4b])**
- **Do the above logics enjoy some nice "partial" witnessed completeness?**

Failure of witnessed completeness & partial witnessed conditions

On the failure of witnessed completeness ($\Box$ case)

Lemma

$KG_\Box + \Box\neg\neg x \to \neg\neg\Box x$, and $MG_\Box + \Box\neg\neg x \to \neg\neg\Box x$ are not complete with respect to witnessed models.

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proof: We can prove that the formula

$$(\Box(((p \to q) \to q) \wedge (q \to p)) \to ((\Box p \to \Box q) \to \Box q)$$

is sound for (valued) witnessed models, but is not a consequence of $KG_{\Box} + \Box\neg\neg x \to \neg\neg\Box x$.

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Hence, the same happens for KG and MG.

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Hence, the same happens for KG and MG.

This implies that the logics are not quasi-witnessed, which was the case in Product logic.

On the failure of witnessed completeness ($\Diamond$ case)

Lemma

$KG_\Diamond + (\Diamond x \to \Diamond y) \to (\neg\Diamond y \vee \Diamond(x \to y))$ is not complete w.r.t. witnessed models.

In the same spirit but more complicated to prove, the formula

$$(\Diamond p \to \Diamond q) \wedge ((\Diamond q \to \Diamond z) \to \Diamond z)) \to (\Diamond z \vee \Diamond((p \to q) \wedge ((q \to z) \to z))$$

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*$\Diamond$-**witnessed** $\iff$ witnessed for $\Diamond$-formulas.

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proof: Over the canonical model from [4], for $v, \Diamond\varphi$ with $v(\Diamond\varphi) = 1$, there must exist w with Rvw and $w(\chi) < w(\varphi)$ for each $v(\Diamond\chi) < 1$.

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Axiomatizing the logics of finite models

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$$W_\Box := (\Box(((x \to y) \to y) \land (y \to x)) \to ((\Box x \to \Box y) \to \Box y)$$

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Theorem

$MG + W_\Box$ is complete with respect to witnessed models.

The proof uses the fact that the canonical model for MG is $\Diamond$-witnessed, and an almost identical proof to the next theorem for the $\Box$ case.

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This gives a world g with Rhg, $g(\phi) \geq g(\varphi) > \alpha$ and $g(\chi) > g(\varphi)$. Let

$$\sigma(x) := \begin{cases} x & \text{if } x > g(\varphi), \\ \frac{x-\beta}{g(\varphi)-\beta}(\alpha - \beta) + \beta & \text{if } x \in (\beta, g(\varphi)], \\ x & \text{otherwise.} \end{cases}$$

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$g' = \sigma \circ g \in W$, and we can prove that Rhg'. Since $g'(\varphi) = \alpha$, this witnesses $\Box\varphi$.

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Let h and $\Diamond\varphi$, and let $0 < \alpha := h(\Diamond\varphi) < 1$ and $\beta := min\{h(\Diamond\psi)\colon h(\Diamond\psi) > \alpha\}$.

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Instantiating $W_\Diamond$ and using that $\Diamond$ distributes over $\vee$, we get that

$$h(\Diamond\chi \vee \Diamond((\phi \to \varphi) \wedge ((\varphi \to \chi) \to \chi))) = 1$$

Since $h(\Diamond\chi) < \alpha$, necessarily $h(\Diamond((\phi \to \varphi) \wedge ((\varphi \to \chi) \to \chi)))) = 1$. We already proved that the model is $\langle\Diamond, 1\rangle$-witnessed there is $g \in W$ with Rhg and $g((\phi \to \varphi) \wedge ((\varphi \to \chi) \to \chi)) = 1$, therefore, $g(\phi) \leq g(\varphi) < \alpha$ and $g(\chi) < g(\varphi)$ (since $g(\chi) < 1$) necessarily).

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$$\sigma(x) \coloneqq \begin{cases} x & \text{if } x < g(\varphi), \\ \frac{\beta - x}{\beta - g(\varphi)}(\beta - \alpha) + \alpha & \text{if } x \in [g(\varphi), \beta), \\ x & \text{otherwise.} \end{cases}$$

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Instantiating $W_\Diamond$ and using that $\Diamond$ distributes over $\vee$, we get that

$$h(\Diamond\chi \vee \Diamond((\phi \to \varphi) \wedge ((\varphi \to \chi) \to \chi))) = 1$$

Since $h(\Diamond\chi) < \alpha$, necessarily $h(\Diamond((\phi \to \varphi) \wedge ((\varphi \to \chi) \to \chi)))) = 1$. We already proved that the model is $\langle\Diamond, 1\rangle$-witnessed there is $g \in W$ with Rhg and $g((\phi \to \varphi) \wedge ((\varphi \to \chi) \to \chi)) = 1$, therefore, $g(\phi) \leq g(\varphi) < \alpha$ and $g(\chi) < g(\varphi)$ (since $g(\chi) < 1$) necessarily). Let

$$\sigma(x) := \begin{cases} x & \text{if } x < g(\varphi), \\ \frac{\beta - x}{\beta - g(\varphi)}(\beta - \alpha) + \alpha & \text{if } x \in [g(\varphi), \beta), \\ x & \text{otherwise.} \end{cases}$$

$g' = \sigma \circ g \in W$, and we can prove that Rhg'. Since $g'(\varphi) = \alpha$, this witnesses $\Diamond\varphi$.

Ongoing/future questions

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  - modal logics over a Gödel set with a finite set of downward/upward accumulation points (in a given order)?

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Dziekuje!

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