A Completeness Theorem for Topological Doctrines

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July 30, 2026

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Beyond Interior Algebras

In 1944, McKinsey and Tarski in a celebrated paper proved that every interior algebra embeds into the interior algebra of the subsets of a topological space. This emedding theorem is in fact a strong completeness theorem for propositional $S4$ modal logic.

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Our aim is to transfer this result from single algebras to whole categories. In other words, given a category E (say with finite limits) whose subobjects semilattices are equipped with interior algebra structures, what are the conditions that E must satisfy in order to be embeddable into (a power of) the category of topological spaces?

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In order to appropriately formulate the problem, we need a closer look to **Top**, the category of topological spaces.

Top as a Category

In the category **Top** of topological spaces and continuous maps, modal operators ($\Diamond$=closure, $\Box$=interior) are applied to subsets of a topological space T; subsets can be viewed as subspaces (using the induced topology) and as such they represent a subclass of the class of monomorphisms (actually, the regular ones).

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The orthogonal class of subspaces is the class of surjective morphisms (which are the epis here) and together surjective morphisms and subspaces form a so-called factorization system. Thus, we shall start from an abstract factorization system $(\mathcal{E}, \mathcal{M})$ with the plan of introducing modalities on the $\mathcal{M}$-part of such a factorization system. This is in strict analogy to what happens in our target category **Top**. Let's start from basic definitions.

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Factorization Systems

Let E be a lex category. An orthogonal factorization system on E can be defined as a pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms of E such that:

- Each morphism of E can be written as em with $e \in \mathcal{E}$ and $m \in \mathcal{M}$
$$X \xrightarrow{e} Y \xrightarrow{m} Z$$
- Both $\mathcal{E}$ and $\mathcal{M}$ contain identities and are closed under left and right composition with isomorphisms.
- Every arrow of $\mathcal{E}$ is orthogonal on the left to every arrow of $\mathcal{M}$. In other words, for each commutative square
$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, dashed] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$
where $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there is a unique diagonal filler $w : B \longrightarrow C$ such that $ew = u$ and $wm = v$.

f-Regular Categories

An orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ is called *stable* if $\mathcal{E}$ and $\mathcal{M}$ are stable under pullbacks. (In fact, $\mathcal{M}$ is necessarily stable under pullbacks in an orthogonal factorization system.)

Definition

An *f-regular category* (short for 'factorization-regular category') is a lex category E equipped with a stable orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{M}$ is included in the class of all monomorphisms, and contains the class of regular monomorphisms.
A lex functor between f-regular categories is *f-regular* when it preserves the factorization system.

$\mathcal{M}$-subobjects

In an f-regular category $(\mathsf{E}, \mathcal{M})$, the arrows in $\mathcal{M}$ with codomain X (better, their equivalence classes under isomorphisms) are called the *$\mathcal{M}$-subobjects* of X and we denote them as $S \overset{s}{\hookrightarrow} X$. We think of S as a 'subspace' of X.

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The $\mathcal{M}$-subobjects of X form an inf-semilattice $Sub_{\mathcal{M}}(X)$; the preorder of such semilattice is as follows: given $s_1 : S_1 \hookrightarrow X$ and $s_2 : S_2 \hookrightarrow X$, we write $S_1 \leq S_2$ if there is $h : S_1 \longrightarrow S_2$ such that $hs_2 = s_1$.

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Actually $Sub_{\mathcal{M}}$ is a contravariant functor with value into the category of inf-semilattices: $Sub_{\mathcal{M}}(f)$ - denoted f^* or f^{-1} - is pullback. More is true indeed.

Lawvere doctrines

Definition

A *regular Lawvere doctrine* is a pair $(\mathsf{E}, \mathcal{P})$ given by a lex category E and a functor $\mathcal{P} : \mathsf{E}^{op} \longrightarrow \mathsf{SMLat}$ such that:

- For every $f : X \longrightarrow Y$ in E, the function $f^* = \mathcal{P}(f) : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ has a left adjoint $\exists_f : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$.
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Proposition

If $(\mathsf{E}, \mathcal{M})$ is an f-regular category, then $(\mathsf{E}, \mathrm{Sub}_{\mathcal{M}})$ is a regular Lawvere doctrine.

Coherent and Boolean Logics

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Definition

An *f-coherent category* is an f-regular category $(\mathsf{E}, \mathcal{M})$ whose functor $\mathrm{Sub}_{\mathcal{M}}$ is valued in the category of distributive lattices.
An f-regular functor between f-coherent categories is *f-coherent* when it moreover preserves finite joins of $\mathcal{M}$-subobjects.
We say that $(\mathsf{E}, \mathcal{M})$ is *f-Boolean* when $\mathrm{Sub}_{\mathcal{M}}$ is valued in the category of Boolean algebras.

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Modal Categories

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Definition

A *modal category* is an f-coherent category $(\mathsf{E}, \mathcal{M})$ such that for every object X, the lattice $\mathrm{Sub}_{\mathcal{M}}(X)$ is a modal lattice. Moreover, the following conditions must be satisfied:

- *(Continuity)* for every $f : X \longrightarrow Y$ and $S \in \mathrm{Sub}_{\mathcal{M}}(Y)$ we have
$$\Diamond f^* S \leq f^* \Diamond S; \tag{1}$$
- *(Subspace)* For every $m : A \hookrightarrow X$ in $\mathcal{M}$ and for every $S \in \mathrm{Sub}_{\mathcal{M}}(A)$ it holds that
$$\Diamond S = m^* \Diamond \exists_m S. \tag{2}$$

Modal Categories: examples

Relational **G**-sets are counterpart frames. Given a graph

$$\mathbf{G} := \{d, c : A \rightrightarrows O\},$$

a *relational* **G**-*set* X consists of the following data:

- for every $\alpha \in O$, a set X_α;
- for every $k \in A$ with $d(k) = \alpha$ and $c(k) = \beta$, a relation $X_k \subseteq X_\alpha \times X_\beta$.

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Again we take as $\mathcal{M}$ the class of regular monos (= collections of subsets with restricted X_k). For $S \in \mathrm{Sub}_r(X)$ and $a \in X_\alpha$, we put

$$a \in (\Diamond S)_\alpha \quad \text{iff} \quad \text{there are } k : \alpha \longrightarrow \beta \text{ and } b \in X_\beta \quad \text{s.t. } akb \text{ and } b \in S_\beta \quad .$$

(we write akb for $(a, b) \in X_k$). In terms of forcing, this reads as

"$a \models_\alpha \Diamond S$ iff for some k-counterpart b of a we have $b \models_\beta S$".

The Representation Theorem

Recall that a *morphism* Φ between f-coherent categories $\Phi : (\mathsf{E}_1, \mathcal{M}_1) \longrightarrow (\mathsf{E}_2, \mathcal{M}_2)$ is a lex functor that preserves factorizations and joins; if $(\mathsf{E}_1, \mathcal{M}_1)$ and $(\mathsf{E}_2, \mathcal{M}_2)$ are modal categories, a *modal morphism* between them is an f-coherent functor $\Phi : (\mathsf{E}_1, \mathcal{M}_1) \longrightarrow (\mathsf{E}_2, \mathcal{M}_2)$ that also preserves the modal operators.

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We say that a modal morphism Φ is a *conservative embedding* when for any $m \in \mathcal{M}_1$, if $\Phi(m)$ is iso, then m is iso as well.

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Theorem (G., Marquès, JSL 2025)

Every small modal category $(\mathsf{E}, \mathcal{M})$ *has a conservative embedding into a modal category of the kind* $\mathbf{Rel}^{\mathbf{G}}$.

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Product Independence

For an embedding theorem into (a power of) **Top**, we need extra conditions. The first condition are the $S4$ axioms:

$$S \leq \Diamond S, \qquad \Diamond\Diamond S \leq \Diamond S \quad . \tag{S4}$$

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Let E be a modal category and let X, Y be two objects of E. The *product independence axiom* states that for all $S \in \mathrm{Sub}_{\mathcal{M}}(X)$ and $T \in \mathrm{Sub}_{\mathcal{M}}(Y)$,

$$(\pi_X^{-1}\Diamond S) \wedge (\pi_Y^{-1}\Diamond T) \leq \Diamond(\pi_X^{-1} S \wedge \pi_Y^{-1} T). \tag{PI}$$

where π_X, π_Y are the two projections of domain $X \times Y$.

Product Independence

Note that the reverse inequality of (PI) holds in any modal category. Once translated in symbolic logic terms, the product independence axiom says that $\Diamond$ distributes over a conjunction $S \wedge T$ in case S and T have disjoint free variables:

$$\Diamond S(x) \wedge \Diamond T(y) \to \Diamond(S(x) \wedge T(y)) \quad .$$

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A remarkable consequence of (PI) is the fact that projections are open, i.e. that we have

$$\pi_X^{-1} \Diamond S \ = \ \Diamond \pi_X^{-1} S$$

for all $S \in \mathrm{Sub}_{\mathcal{M}}(X)$. This fact, as we shall see, facilitates the construction of the corresponding logical calculi.

Internal Relations and Partial Maps

We still need a condition for the representation theorem into **Top**. An (internal) relation in a modal category $(\mathsf{E}, \mathcal{M})$ is an $\mathcal{M}$-subobject of a cartesian product $R \subseteq X \times Y$.

Among relations, we have the graphs of partial maps. A *partial map* f from X to Y is given by a $\mathcal{M}$-subobject $d_f : \mathrm{dom}(f) \hookrightarrow X$, and a morphism $t_f : \mathrm{dom}(f) \to Y$. The *graph* of f is $(d_f, t_f) : \mathrm{dom}(f) \hookrightarrow X \times Y$ (which is a relation). A partial map from X to Y is denoted

$$X \rightharpoonup Y$$

Given a relation $R \subseteq X \times Y$ and $A \in Sub_{\mathcal{M}}(X)$, we denote by $RA = \exists_{\pi_Y}(R \wedge \pi_X^{-1}A)$ the direct image of $A \subseteq X$ under R and we denote by $R^{-1} \subseteq Y \times X$ the reciprocal relation.

The Loop Contraction Condition

A *loop* in a modal category is a sequence of binary relations $R_1, \ldots, R_n$ such that the composite $R_1 \circ \cdots \circ R_n$ makes sense (i.e. is type-matching) and such that the codomain of R_1 coincides with the domain of R_n. The object which is both the codomain of R_1 and the domain of R_n is called the *anchor* of the loop.

To each loop $R_1, \ldots, R_n$, we associate the following "loop contraction" condition, where A ranges over the subobjects of the anchor:

$$\Diamond R_1 \Diamond R_2 \Diamond \cdots \Diamond R_n A \leq \Diamond A. \tag{LC}$$

We shall identify some loops satisfying this condition in **Top** and call them 'acceptable'. We first give an example of such a loop: take

$$f_1 : X_1 \rightharpoonup Y_1, \qquad f_2 : X_2 \rightharpoonup Y_2$$

Loop contraction principle: example

$$\begin{tikzcd}[column sep=small]
 & X_1 \times X_2 \arrow[dl, "f_1 \times \mathrm{id}"'] \arrow[dr, "\mathrm{id} \times f_2"] & \\
Y_1 \times X_2 \arrow[dr, "\mathrm{id} \times f_2"'] & & X_1 \times Y_2 \arrow[dl, "f_1 \times \mathrm{id}"] \\
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\end{tikzcd}
$$

Acceptable Loops

We define by induction the *acceptable* loops:

1. the empty loop is acceptable (on any anchor),
2. identities can be added anywhere in acceptable loops to produce new acceptable loops,
3. if w and w' are acceptable, then w, w' is acceptable,
4. if w is acceptable and if f is a partial map, then f, w, f^{-1} is acceptable,
5. if $R_1, \ldots, R_n$ and $S_1, \ldots, S_n$ are acceptable, then $R_1 \times S_1, \ldots, R_n \times S_n$ is acceptable.

Topological Representation

Theorem (G., Marquès)

A Boolean modal category is conservatively embeddable in a power of **Top** *iff it is topological i.e. iff it satisfies the following conditions:*

1. (S4) *axioms for all subobjects;*
2. *product independence axiom;*
3. *loop contraction axiom for all acceptable loops.*

(embeddability into **Top** *requires in in addition that $Sub_{\mathcal{M}}(\mathbf{1})$ is the two-element Boolean algebra).*

The loop contraction axiom is redundant for the acceptable loops entirely built up from partial maps which are total.

Topologies involved in the proof

Alexandrov spaces (induced by preordered) sets are not sufficient for the above result: we need to generalize them.

Take a diagram of sets

$$X_1 \longrightarrow X_2 \longrightarrow \cdots X_i \longrightarrow \cdots$$

and relations $R_{ij} \subseteq X_i \times X_j$ for $i \leq j$ such that

- R_{ij} contains the graph of $X_i \to \cdots \to X_j$
- $R_{ij} R_{jk} \subseteq R_{ik}$

These data leads to a topology on $\varinjlim_i X_i$ defined as follows

Topologies involved in the proof

For $x \in \varinjlim_i X_i$, a basis of open neighborhoods is given by the sets

$$I_p \ := \ \{ \, [q] \ | \ pR_{ij}q \, \}$$

where $x = [p]$ and $p \in X_i$ (in other words, we have $y \in I_p$ iff there exist j, $q \in X_j$ such that $y = [q]$ and $pR_{ij}q$).

Examples:

- Posets (take the constant sequence).
- Metric spaces: take $X_i := X$ for all i and $pR_{ij}q \leftrightarrow d(p,q) \leq 1/i - 1/j$.

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Building a Logical Calculus

We point out that our representation theorem can be stated in terms of Lawvere (hyper)doctrines as well: in fact, our modal categories can be seen as modal doctrines with comprehension. It is indeed not difficult to built a modal (topological) category out of a modal (topological) doctrine: details can be found in the last section of our arXiv preprint.

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Building syntactic calculi corresponding to our modal categories requires some care, because modal operators are not pullback stable, they are only 'half-stable' (see the continuity condition in the definition of a modal category). In our JSL paper, a solution based on a restricted form of λ-abstraction is adopted (this is similar to a solution known from long time in the first-order modal logic literature oriented towards philosophical applications).

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A simplified solution is possible for the topological semantics considered here, because of the above mentioned fact that projections are open.

Building a Logical Calculus

Symbols and formulae should have a direct categorical meaning.

Sort	**Topological space**
Function symbol	**Continuous function**
Formula	**Subspace** ($\neq$ monomorphism)
Disjunction	Union
Negation	Complement
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$\exists$	Direct image
$\Diamond$	**Closure**
$\Box$	**Interior**

Building a Logical Calculus

Formulae like

$$\Diamond P(x) \wedge Q(x, y)$$

are not problematic because they can be read indifferently as

$$\pi_X^{-1}(\Diamond P) \cap Q$$

or as

$$\Diamond(\pi_X^{-1} P) \cap Q \quad .$$

Recall in fact that projections are open in **Top** (this is not the case for relational G-Sets).
Still, there are problems with substitutions.

Modalities and Substitution

Classical first-order logic: Substitution commutes with the other operations.

$$\varphi(x)[f(z)/x] = \varphi(f(z)/x)$$

Topological modal first-order logic: Substitution **does not** commute with $\Diamond$ and $\Box$.

Reason: $\Diamond f^{-1}(\varphi) \subseteq f^{-1}(\Diamond\varphi)$ when $f : X \to Y$ is continuous and $\varphi \subseteq Y$.
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- $[\![(\Box(x = y))[z/x, z/y]]\!] = \{p \in X \text{ isolated}\}$

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$$\varphi(x)[f(z)/x] = \varphi(f(z)/x)$$

Topological modal first-order logic: Substitution **does not** commute with $\Diamond$ and $\Box$.

Reason: $\Diamond f^{-1}(\varphi) \subseteq f^{-1}(\Diamond\varphi)$ when $f : X \to Y$ is continuous and $\varphi \subseteq Y$.
$\Diamond f^{-1}(\varphi) \supseteq f^{-1}(\Diamond\varphi)$ when f is *open*.

Example: $\Delta : X \to X \times X$ is not open.

- $[\![x = y]\!] = \{(p, p) \mid p \in X\}$
- $[\![\Box(x = y)]\!] = \{(p, p) \mid p \in X \text{ isolated}\}$
- $[\![(\Box(x = y))[z/x, z/y]]\!] = \{p \in X \text{ isolated}\}$
- But $[\![\Box(z = z)]\!] = \{p \in X\}$

Modalities and Substitution

The most radical solution is to consider substitution as a logical operator (like connectives and quantifiers). In topological semantics, one can simply redefine the inductive step of substitution concerning modal operators as

$$(\Diamond\varphi)[t_1/x_1, \ldots, t_n/x_n] := \exists x_1 \cdots \exists x_n \, (t_1 = x_1 \wedge \cdots \wedge t_n = x_n \wedge \Diamond\varphi) \quad .$$

All the axioms we need can be expressed in this way, in particular continuity becomes

$$\Diamond(\varphi[t_1/x_1, \ldots, t_n/x_n]) \to (\Diamond\varphi)[t_1/x_1, \ldots, t_n/x_n] \quad .$$

Some Research Directions

- Topological semantics is expressive: important classes of spaces (compact spaces, Alexandrov spaces, etc.) can be distinguished from generic topological spaces. Axiomatizing them could be an interesting challenging task.

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Some Research Directions

- Topological semantics is expressive: important classes of spaces (compact spaces, Alexandrov spaces, etc.) can be distinguished from generic topological spaces. Axiomatizing them could be an interesting challenging task.
- Developing a model theory specific for topological first order modal logic looks an interesting direction for mathematical applications.
- **Top** does not have good categorical properties, but other modal categories (like relational G-sets which are a quasi-topos) have and might be interesting for higher order modal logic.

References

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THANKS FOR ATTENTION !

Investigating Definability

In a logical category, a relation which is functional in the internal language corresponds to an arrow (we may call this ‘functional comprehension’).

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This is not so in a modal category, because we need continuity as an additional requirement.

This problem involves not only the notion of a function, but also the notion of a subspace and of an isomorphism. We investigate these three cases separately.

Subspace Saturation

Let us start by analyzing the notion of a subspace, i.e. of an arrow $m : A \hookrightarrow X$ belonging to $\mathcal{M}$. We know that m must satisfy condition (2), namely we need

$$\Diamond S = m^* \Diamond \exists_m S.$$

for every $S \in \mathrm{Sub}_{\mathcal{M}}(A)$.
We wonder what happens if a converse is satisfied: namely, is it sufficient that (2) holds for all $S \in \mathrm{Sub}_{\mathcal{M}}(A)$ for a *monomorphism* m to be in $\mathcal{M}$?
If we look at our reference semantic category $\mathrm{Rel}^{\mathbf{G}}$, we see that this is not the case, in the sense that we must make the condition 'stable' by adding a parameter Z for this to be true.

Subspace Saturation

We say that a mono $m : A \hookrightarrow X$ is a *brittle subspace* iff for all Z, for all $S \in \mathrm{Sub}_{\mathcal{M}}(A \times Z)$ we have that

$$\Diamond S = (m \times 1_Z)^* \Diamond \exists_{m \times 1_Z} S. \tag{3}$$

In $\mathrm{Rel}^{\mathbf{G}}$, the $\mathcal{M}$-subobjects can then be characterized as the brittle subspaces.

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Subspace saturation principle (SS). *A modal category satisfies the subspace saturation principle iff every brittle subspace is a subspace (i.e. it belongs to $\mathcal{M}$).*

Functional Saturation

A *functional relation* is some $R \overset{r}{\hookrightarrow} X \times Y$ in $\mathcal{M}$ such that the composite map

$$R \overset{r}{\hookrightarrow} X \times Y \xrightarrow{\pi_X} X$$

is a monomorphism and is also in $\mathcal{M}^{\perp}$ (this can be reformulated in the customary way using the internal language of doctrines).

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A *continuous relation* on the other hand is some $R \overset{r}{\hookrightarrow} X \times Y$ in $\mathcal{M}$ such that for all Z, for all $S \in \mathrm{Sub}_{\mathcal{M}}(Y \times Z)$ we have that the following inequality holds in $Sub_{\mathcal{M}}(X \times Y \times Z)$

$$\Diamond(R \circ S) \leq R \circ (\Diamond S) \tag{4}$$

(here $\circ$ is defined in the expected way).

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Functional saturation principle (FS). *A modal category satisfies the functional saturation principle iff every brittle morphism $R \hookrightarrow X \times Y$ is the graph of an arrow, meaning that there is $h : X \longrightarrow Y$ such that we have $(1_X, h) \simeq R$ in $\mathrm{Sub}_{\mathcal{M}}(X \times Y)$.*

Isomorphism Saturation

We say that $h : X \longrightarrow Y$ is *open* iff it satisfies the condition

$$h^*(\Diamond S) \leq \Diamond(h^* S)$$

for every $S \in \mathrm{Sub}_{\mathcal{M}}(Y)$; it is *stably open* iff $h \times 1_Z$ is open for every Z.

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We say that h is a *brittle isomorphism* iff it is mono, it is in $\mathcal{M}^\perp$ and it is also stably open. We can now formulate the

Isomorphism saturation principle (IS). *A modal category satisfies the isomorphism saturation principle iff every brittle isomorphism is an isomorphism.*

Saturated Modal Categories

Theorem

The above saturation principle are all equivalent in a modal category.

We say that a modal category is *saturated* iff it satisfies one of (hence all) the saturation principles (SS),(FS),(IS).