The Eliminability of Delta in Gödel Logics[1]

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Propositional Gödel logic

Usual propositional language, countably many propositional variables A_i, connectives $\wedge, \vee, \to$, constants $\bot, \top$.
Truth-value set $\{0,1\} \subseteq V \subseteq [0,1]$, closed.
Abbreviations $\neg A := A \to \bot$, $\top := \bot \to \bot$.

Connectives

$$\mathcal{I}(A \wedge B) = \min(\mathcal{I}(A), \mathcal{I}(B)), \quad \mathcal{I}(A \vee B) = \max(\mathcal{I}(A), \mathcal{I}(B)).$$

$$\mathcal{I}(A \to B) = \begin{cases} \mathcal{I}(B) & \text{if } \mathcal{I}(A) > \mathcal{I}(B), \\ 1 & \text{if } \mathcal{I}(A) \leq \mathcal{I}(B). \end{cases}$$

$$\mathcal{I}(\neg A) = \begin{cases} 0 & \text{if } \mathcal{I}(A) > 0 \\ 1 & \text{otherwise} \end{cases}$$

Why Define a Logic by Its Valid Sentences?

- Do not depend merely on the number of truth values.
- Depend on the order-topological structure of the truth-value set.
- The classification of first-order Gödel logics highly non-trivial.
- We *cannot* characterize a logic purely by its truth-value structure V (too many "representations").

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$$\mathbf{G}_V = \{A : \forall \mathcal{I} \text{ into } V : \mathcal{I}(A) = 1\}$$

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Issue: Different value-sets $V \subseteq [0,1]$ can induce the *same* G_V. e.g.

$$V_\uparrow = \{\,1 - \tfrac{1}{n} : n \geq 1\} \cup \{1\} \quad \text{and} \quad V'_\uparrow = \{\,1 - \tfrac{2}{n} : n \geq 1\} \cup \{1\}$$

both yield $G_\uparrow$.

This ensures a *one-to-one correspondence* between:

$$\{\text{distinct Gödel logics}\} \longleftrightarrow \{\text{sets of formulas } G_V\}.$$

Examples

$$
\begin{array}{ll}
V = \{0,1\} & \to \mathbf{G}_V = CPL \\
V_1 = \{0, 1/2, 1\}, V_2 = \{0, 1/3, 1\} & \to \mathbf{G}_{V_1} = \mathbf{G}_{V_2} \\
V_\uparrow = \{1 - 1/n : n \geq 1\} \cup \{1\} & \to \mathbf{G}_{V_\uparrow} = \mathbf{G}_\uparrow \\
V_\downarrow = \{1/n : n \geq 1\} \cup \{0\} & \to \mathbf{G}_{V_\downarrow} = \mathbf{G}_\downarrow \\
V_m = \{1\} \cup \{1 - 1/k : 1 \geq k \geq m-1\} & \to \mathbf{G}_{V_m} = \mathbf{G}_m
\end{array}
$$

In fact, there are *uncountably many* closed subsets of $[0,1]$ but only *countably many* distinct propositional Gödel logics.

The asymmetry of 0 and 1

But there is a hidden asymmetry.

$$0 \qquad \rule{10cm}{0.4pt} \qquad 1$$

Negation detects whether a formula has value 0:

$$\neg A = 1 \quad \text{iff} \quad A = 0.$$

But the standard connectives cannot detect whether

$$A = 1$$

exactly.

They cannot separate 1 from values arbitrarily close to 1.

The Δ operator (Baaz)

$$\mathcal{I}(\Delta A) = \begin{cases} 1 & \text{if } \mathcal{I}(A) = 1 \\ 0 & \text{if } \mathcal{I}(A) < 1 \end{cases}$$

So ΔA does not ask:

“How true is A?”

It asks:

“Is A absolutely true?”

Proposition: Δ is not definable with other connectives and variables.

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Proof: There are a finite number of 1-variable functions in Gödel logic:

$$\top, \bot, A, \neg A, \neg A \vee A, \neg A \supset A.$$

Assume Δ is definable by some of the function F, i.e.,

$$\Delta(A) \leftrightarrow F(A).$$

Check F in G_3, because if Δ is not definable in G_3 then it is not definable in all larger Propositional $\mathbf{G}_V$.
None of them defines Δ and
they are closed under composition by all connectives:

A	$\neg A$	$\top$	$\bot$	$A \vee \neg A$	$\neg A \to A$	ΔA
0	1	1	0	1	0	0
1/2	0	1	0	1/2	1	0
1	0	1	0	1	1	1

Axiomatization of Gödel logics with Δ

Baaz introduced and axiomatised in the context of Gödel logics using Hilbert style calculus [1]:

$\Delta 1$	$\Delta A \vee \neg \Delta A$	$\Delta 2$	$\Delta(A \vee B) \to (\Delta A \vee \Delta B)$
$\Delta 3$	$\Delta A \to A$	$\Delta 4$	$\Delta A \to \Delta\Delta A$
$\Delta 5$	$\Delta(A \to B) \to (\Delta A \to \Delta B)$		$A \vdash \Delta A$

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Then $G^{\Delta}_{[0,1]}$ is axiomatized by $G_{[0,1]} + \Delta$.

Δ is useful, but problematic

Why Δ is useful

- detects absolute truth;
- restores the missing symmetry at 1;
- gives extra expressive power;
- has natural modal/stability readings.

Why Δ is difficult

- not definable by standard connectives;
- breaks substitution closure;
- affects the deduction theorem;
- complicates proof theory and cut elimination;
- equivalence principle fails;
- is not intermediate logic anymore.

Why Elimination Theorems Matter

Elimination results are central in proof theory.

- cut elimination
- Skolemization and deSkolemization
- quantifier elimination
- normal forms

They usually tell us that some extra structure is useful for proofs, but not always essential for expressivity.

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Why Eliminate Δ?

- Simplifies semantic and proof–theoretic analysis
- Simplifies the study of prenex fragments
- Restores substitution–closure (lost once Δ is added)
- Influences axiomatizations and decision procedures

Question

Can we keep the expressive power of the absoluteness operator

without keeping the operator itself?

In other words:

$\boxed{\text{When can } \Delta \text{ be eliminated from Gödel logic?}}$

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This tension motivated our recent paper

(Matthias Baaz, Mariami Gamsakhurdia:) **Goedel Logics: On the Elimination of The Absoluteness Operator**- TBiLLC2025 Postproceedings.

The problem

Can we eliminate Δ?

More precisely:

- Can every formula with Δ be transformed into a formula without Δ?
- Does the transformation preserve validity?
- Does it preserve logical equivalence?
- What changes in the first-order case?

Two notions of elimination

1. Validity-equivalent elimination

$$\models A \quad \text{iff} \quad \models A^T,$$

where A^T is Δ-free.

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2. Logically-equivalent elimination

$$A \equiv A^T.$$

This is much stronger: the two formulas have the same truth value under every interpretation.

The main idea

Instead of attacking Δ directly, we slightly restrict the semantics.

$$\boxed{\text{Atomic formulas are never allowed to take value } 1.}$$

Only the truth constant $\top$ may have value 1.

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Only the truth constant $\top$ may have value 1.

Formally, under **restricted semantics**:

$$I(P) < 1$$

for every atomic formula $P \neq \top$.

We denote the corresponding logic by $\mathbf{G}_V^-$.

Why does restricted semantics help?

In standard semantics, Δ can distinguish:

$$A = 1 \qquad \text{from} \qquad A < 1.$$

Under restricted semantics, atoms never reach 1.

So for atomic formulas:

$$\Delta P = 0.$$

The only atomic formula recognized by Δ as absolutely true is $\top$.

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Consequence: Certain **substitution properties fail**,
but Δ becomes amenable to elimination:
As $\neg\Delta P$ is true for any variable, $\neg\Delta\top$ is false.

Theorem: Reduction to standard semantics

Restricted semantics can be encoded inside ordinary semantics with Δ.

For a propositional formula $F(X_1, \ldots, X_n)$:

$$F \text{ is valid under restricted semantics}$$

iff

$$(\neg\Delta X_1 \wedge \cdots \wedge \neg\Delta X_n) \to F$$

is valid under standard semantics with Δ.

The antecedent simply says:

none of the variables has value 1.

Equivalence Principle

$$A \leftrightarrow B \Rightarrow E(A) \leftrightarrow E(B)$$

The specific case

$$A \leftrightarrow B \Rightarrow \Delta(A) \leftrightarrow \Delta(B)$$

fails in the restricted semantics.

Example

$$\mathcal{I}(A) = 1,\ 0 < \mathcal{I}(B) < 1$$

$$\mathcal{I}(\Delta A) = 1,\ \mathcal{I}(\Delta B) = 0$$

$\mathcal{I}(A \leftrightarrow B)$ is not 0.

Lemma (Context-Closure)

For any formulas A, B, C, D, and any context function E, the following implication holds: from $A \vee B \wedge (C \leftrightarrow D)$ we derive $A \vee B \wedge (E(C) \leftrightarrow E(D))$

Lemma (Transitivity)

The transitive closure of equivalence in the restricted semantics holds, i.e., for any formulas C, D, L:

$$(A \vee (B \wedge (C \leftrightarrow D))) \wedge (D \leftrightarrow L) \Rightarrow (A \vee B \wedge (C \leftrightarrow L)).$$

Lemma (Local Δ–Elimination)

Given any expression $\triangle(C \vee (D \wedge a))$ for some variable a in the restricted semantics where C, D are valid expressions we can eliminate $\triangle$ obtaining C

$$\frac{\triangle(C \vee (D \wedge a))}{C}$$

Chains replace valuations by order types

Instead of considering infinitely many truth values, we consider finitely many possible order patterns among the variables.

Example:

$$\bot < a < b < \top,$$

$$\bot < b < a < \top,$$

$$\bot < a \leftrightarrow b < \top.$$

The actual values do not matter as much as their ordering.

- Every chain describes an order type of the variables X.
- For each valuation there is a chain that matches the ordering of truth-values assigned to atoms.

Chains: the proof-theoretic intuition

Every Gödel valuation induces a linear ordering of truth values.

A **chain** is a formula describing one such ordering relative to the constants $\top$ and $\bot$, described by positions of variables.

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A **chain** C over the set of propositional variables $X = [X_1, ...X_n]$ is an expression

$$\bot \rhd_1 X_{\Pi(1)} \rhd_2 \top \rhd_1 X_{\Pi(2)} \rhd_3 \cdots \rhd_n X_{\Pi(n)} \rhd_{n+1} \top$$

Π is permutation on $[1 \dots n]$ and $\rhd_i \in \{<, \leftrightarrow\}$.

Chains play for Gödel logic a role similar to **disjunctive normal forms** in classical logic.

Restricted chains

In restricted semantics, no variable may be equivalent to $\top$.

So we allow chains such as

$$\perp < a < b < \top,$$

but not chains such as

$$\perp < a < b \leftrightarrow \top.$$

We denote a chain in the restricted semantics by C^{-}.

This small change is what makes **logically-equivalent elimination** possible at the **propositional** level.

Main (Propositional) Result

Every propositional formula containing Δ is equivalent, under restricted semantics, to a disjunction of Δ-free restricted chains.

Thus Δ is eliminable propositionally under restricted semantics.

Example: the intuition

Consider a formula of the form

$$F := a \vee \Delta(a \vee \neg a).$$

The corresponding chain decomposition yields three chains in standard semantics:

$$(\bot \leftrightarrow a) < \top, (\bot < a) < \top, (\bot < \top \leftrightarrow a)$$

for some variable a.

Example: the intuition

Consider a formula of the form

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The corresponding chain decomposition yields three chains in standard semantics:

$$(\bot \leftrightarrow a) < \top, (\bot < a) < \top, (\bot < \top \leftrightarrow a)$$

for some variable a.
Under restricted semantics, the relevant chains are only:

$$\bot \leftrightarrow a < \top, \qquad \bot < a < \top.$$

The chain where $a \leftrightarrow \top$ is excluded.

The CNF in the restricted semantics
$(\bot \leftrightarrow a) < \top \wedge F \vee (\bot < a) < \top \wedge F$.
evaluate from the innermost first.

Evaluation of the first chain:	**Evaluation of the second chain:**
$a \vee \Delta(a \vee \top)$	$a \vee \Delta(a \vee \bot)$
$a \vee \Delta\top$	$a \vee \Delta a$
$a \vee \top$	$a \vee \bot$
$\top$	a
	$\bot$

After evaluation along the restricted chains, the formula becomes Δ-free.

$$a \vee \Delta(a \vee a \to \bot) \leftrightarrow \neg a.$$

Substitution and Restricted Semantics

Assume we substitute $\top$ for a, we obtain

$$\top \vee \Delta(\top \vee \top \to \bot) \leftrightarrow \neg\top$$

and consequently $\top \leftrightarrow \bot$.

The restricted semantics is not closed under substitution.

Standard vs restricted semantics

Standard semantics	**Restricted semantics**
Atoms may be 1	Atoms are always < 1
Δ detects atomic truth	$\Delta P = 0$ for atoms P
Substitution closure holds without Δ	Substitution closure may fail
Δ not logically eliminable	Δ logically eliminable propositionally

The restriction is small, but its proof-theoretic consequences are large.

Without Δ: standard and restricted coincide

For propositional formulas without Δ:

$$\mathbf{G}_{[0,1]} \models A \quad \text{iff} \quad \mathbf{G}^{-}_{[0,1]} \models A.$$

Why?

Without Δ, formulas cannot distinguish exact truth from values arbitrarily close to truth.

So the restriction only matters when Δ is present.

First-Order Limitations

Full characterization of Axiomatizability

Theorem (Baaz, Preining)

A first-order Gödel logic G_V is recursively enumerable iff one of the following conditions is satisfied:
1. V is finite,
2. V is uncountable and 0 is an isolated point,
3. V is uncountable, and every neighbourhood of 0 is in the prefect subset.

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Not recursively enumerable

- countably infinite truth value set
- every neighbourhood of 0 is countably infinite

Full characterization of Δ-Axiomatizability

Recursively axiomatizable

- finitely valued
- V is uncountable and 0 is an isolated point, and 1 is in the perfect subset,
- V is uncountable, and 0 is in the prefect subset, and 1 is isolated point,
- V is uncountable and both 0 and 1 are isolated points,
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Full characterization of Δ-Axiomatizability

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Not recursively enumerable

- countably infinite truth value set
- either 0 or 1 is not isolated but not in the perfect kernel

First-Order Limitations

The argument used for the propositional case does not extend to the first-order case.

- Case: 1 is not isolated, not in a perfect set.
  0 is isolated or in a perfect set $\to\to$
  the first-order $\mathbf{G}_V^\Delta$ is not recursively enumerable,
  while the first-order $\mathbf{G}_V$ is.
  This holds both for standard and restricted semantics
- Therefore, there is not even an effective validity equivalence elimination of Δ, $\to$ no valid equivalence as in the propositional case.

What changes in first-order logic?

In first-order Gödel logic, quantifiers are interpreted by infima and suprema:

$$I(\forall x A(x)) = \inf_u I(A(u)),$$
$$I(\exists x A(x)) = \sup_u I(A(u)).$$

This creates new problems.

The value 1 may be approached without ever being attained.

Witnessed semantics

Witnessed semantics removes this pathology.

Instead of arbitrary infima and suprema, we require extrema to be attained:

$$I(\forall x A(x)) = \min_u I(A(u)),$$

$$I(\exists x A(x)) = \max_u I(A(u)).$$

Then existential truth has an actual witness.

This improves the interaction between quantifiers and Δ.

$$\text{Infimum is attained:} \quad \exists x\,(A(x) \to \forall y\, A(y)),$$

$$\text{Supremum is attained:} \quad \exists x\,(\exists y\, A(y) \to A(x)).$$

$$(\forall x\, A(x) \to B) \to \exists x\,(A(x) \to B) \quad (A \to \exists x\, B(x)) \to \exists x\,(A \to B(x)).$$

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$$(\forall x\, A(x) \to B) \to \exists x\,(A(x) \to B) \quad (A \to \exists x\, B(x)) \to \exists x\,(A \to B(x)).$$

$$\Delta \forall x\, A(x) \;\leftrightarrow\; \forall x\, \Delta A(x)$$

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$$\Delta\forall x\,A(x) \;\leftrightarrow\; \forall x\,\Delta A(x)$$

In contrast, the existential quantifier exhibits an asymmetry:

$$\exists x\,\Delta A(x) \;\to\; \Delta\exists x\,A(x),$$

Converse implication fails in general.

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$GW^{\Delta}_{[0,1]}$ is axiomatized by

$$GW_{[0,1]} + \Delta + \exists x\, \Delta A(x) \leftrightarrow \Delta \exists x\, A(x).$$

Conceptual conclusion

Setting	**Behavior of Δ**
Propositional, standard	not logically eliminable
Propositional, restricted	logically eliminable
First-order, standard	generally not eliminable
First-order, witnessed	validity-equivalent elimination
Restricted + witnessed	stronger elimination behavior

- Restricted semantics trades substitution-closure for Δ-elimination.
- Under restricted semantics every propositional formula with Δ has a Δ-free equivalent (disjunction of chains).

Conclusions & Open Questions

- Extend to first-order logic when 1 is isolated
- Complexity of decision procedures in G_V^-?
- Can the first-order restricted case be fully classified?
- Are similar restrictions useful for other intermediate or modal logics?
- What are the consequences for hypersequent calculi and interpolation?

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Conjecture: In the restricted semantics all uncountable infinite-valued G_V^- coincide with $G_{[0,1]}^-$ and all countable witnessed infinite-valued G_V^- are not r.e.

In G_f where 1 is not isolated, not in perfect set and 0 is in the perfect set, there is no effective validity-equivalence translation between $G_V^{\triangle -}$ and G_V^-.

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Thank You!

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