Localic distances and asymmetric metrization theorems

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— joint work with A. Pultr and M. Sioen

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SOME MOTIVATION

J. Isbell (1972): defined a locale as *metrizable* if it admits a countably generated uniformity.

A. Pultr (1984): this is equivalent to a genuinely metric point-free structure, given by a *metric diameter*.

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- [S. Henry, Localic metric spaces and the localic Gelfand duality Advances in Mathematics 294, 634–688 **(2016)**]

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Star-additivity: $\forall a \in L, \forall S \subseteq L$ such that $a \wedge s \neq 0$ for all $s \in S$,
(extension of (D3)) $\delta(a \vee \bigvee S) \leq \delta(a) + \sup\{\delta(x) + \delta(y) \mid x, y \in S\}$.

- Then $\{U_p^\delta \mid p > 0\}$ is a basis for a countable uniformity $\mathcal{U}_\delta$.

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frame $\mathfrak{L}(\mathbb{R}_+^\infty)$ presented by

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$$\boxed{\begin{aligned}&\bigvee\{a \otimes b \mid (a,b) \in E\} \circ \bigvee\{c \otimes d \mid (c,d) \in F\} \\ &\quad = \bigcup\{a \otimes b \mid (a,b) \in E\} \circ \bigcup\{c \otimes d \mid (c,d) \in F\}\end{aligned}}$$

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Uniform locale $(L, \mathscr{E})$: non-void filter $\mathscr{E}$ of entourages such that

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- (E2) $\forall E \in \mathscr{E}\; \exists F \in \mathscr{E} : F \circ F \subseteq E$.
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Metric distances $\rightsquigarrow$ countable entourage uniformities

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Hence

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(M2) $\forall p,q \in \mathbb{Q}_+,\ \Delta_p^d \circ \Delta_q^d \subseteq \Delta_{p+q}^d$.

(M3) $\forall p \in \mathbb{Q}_+,\ (\Delta_p^d)^{-1} = \Delta_p^d$.

Distance on L: if it is *admissible* on L, that is,

$$\forall a \in L,\ a = \bigvee \{b \mid b \lhd_d a\}$$

where $\quad b \lhd_d a \equiv \boxed{\exists p \in \mathbb{Q}_+ \colon \Delta_p^d \circ (b \otimes b) \subseteq a \otimes a}$

Diameters ⇝ distances

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The correspondence $\boxed{\delta \rightsquigarrow d_\delta}$ is monotone. $\quad (d_\delta \leq d_{\delta'} \;\equiv\; h_{\delta'} \leq h_\delta \;\Leftrightarrow\; E_p^{\delta'} \subseteq E_p^\delta, \forall p)$

Distances $\rightsquigarrow$ diameters

Let $d\colon L \otimes L \to \mathfrak{L}(\mathbb{R}_+^\infty)$ be a distance on L. Define $\delta_d\colon L \to [0, +\infty]$ by

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$$E_p^{\delta_d} \subseteq \Delta_p^d \subseteq E_{2p+\varepsilon}^{\delta_d} \text{ for every } p \in \mathbb{Q}_+ \text{ and } \varepsilon > 0.$$

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Proposition.

(1) For each metric distance d, $d \leq d_{\delta_d}$.

(2) For each diameter δ, $\delta_{d_\delta} \leq \delta$.

Galois adjunction: Diameters $\leftrightsquigarrow$ distances

Corollary. $\delta_d \leq \delta$ if and only if $d \leq d_\delta$.

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Dist(L): the poset of (admissible) metric distances in L

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Theorem. The maps $\Phi_L = (d \mapsto \delta_d) : \mathrm{Dist}(L) \to \mathrm{Diam}(L)$ and $\Psi_L = (\delta \mapsto d_\delta) : \mathrm{Diam}(L) \to \mathrm{Dist}(L)$ are in a Galois adjunction $\Phi_L \dashv \Psi_L$.

Moreover:

$$\boxed{3\, d_{\delta_d} \leq d \leq d_{\delta_d}} \qquad (nd)^*[0,p) = \Delta^d_{np}$$

$$\boxed{\delta_{d_\delta} \leq \delta < 3\, \delta_{d_\delta}} \qquad (\text{since } x \otimes x \subseteq E^\delta_p \Rightarrow \delta(x) < 3p)$$

Morphisms

Diametric locales $(L, \delta_L), (M, \delta_M)$:
a localic map $f\colon L \longrightarrow M$ is *nonexpansive* [Pultr] if
$(\forall \varepsilon > 0)\ (\forall a \in L\colon \delta_L(a) < \varepsilon)\ (\exists b \in M\colon a \leq f^*(b), \delta_M(b) < \varepsilon)$.

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Lemma. For any cover U, if $x \otimes y \subseteq \bigvee\{u \otimes u \mid u \in U\}$ with $y \neq 0$, then $x \le Uy$.

The non-symmetric setting: quasi-metrics and quasi-uniformities

quasi-uniform locale: drop the symmetry axiom $E \in \mathscr{E} \;\Rightarrow\; E^{-1} \in \mathscr{E}$.

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The admissibility condition $(*)$ is equivalent to saying that

$$\boxed{\text{the triple } (L, L_1(\mathcal{E}), L_2(\mathcal{E})) \text{ is a } \textit{biframe}}$$

Quasi-metrization theorems

quasi-distance: drop the symmetry axiom and extend admissibility to $\overline{\mathcal{E}_d}$.

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Theorem 1. Let $(L, \mathcal{E})$ be a quasi-uniform frame. Then $\mathcal{E}$ has a countable basis if and only if there is a quasi-distance d on L such that $\mathcal{E} = \mathcal{E}_d$.

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Theorem 2. Let (L, L_1, L_2) be a regular biframe. Then (L, L_1, L_2) admits a countably generated quasi-uniformity if and only if it admits a countably generated quasi-nearness.

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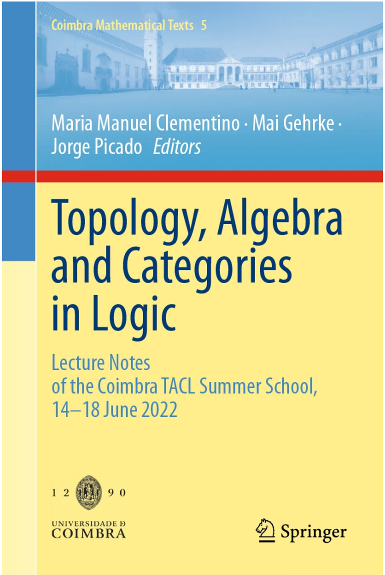

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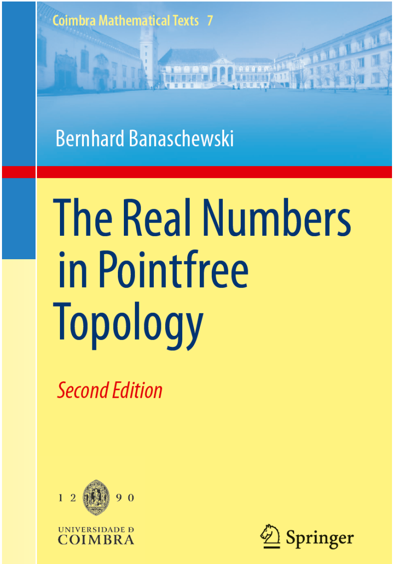

published in memory of
Bernhard Banaschewski
(March 22, 1926 - October 31, 2022),
in the centenary year of his birth

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