Topological and Differentiable Aspects of Clifford semigroups

Giuseppe Zecchini (j.w.w. Stefano Bonzio & Andrea Loi)

TACL 2026, 31 July 2026

University of Cagliari

Introduction

Clifford semigroup $\equiv$ **Pasting** of its **maximal subgroups** along its (meet) **semilattice of idempotents** via a family of **transition homomorphisms**.

Goal of the paper

Interplay between global topological and differentiable properties of Clifford semigroup and that of its components.

Introduction

arXiv preprint $\longrightarrow$

Three central themes:

1. **Topologies on a Clifford semigroup**.
   - Disjoint union topology (*strong semilattice of topological groups*) isolates fibers (*discreteness of idempotents*);
   - Compact Hausdorff case (Yeager & Bowman topologies);
   - Metrizability (Bokalo conjecture);
   - Constructive solution (Bowman case).
2. **Hilbert fifth problem for Clifford semigroups**. When all the maximal subgroups are Lie groups?
3. **A rigidity result**. C^1 regularity at idempotents implies strong semilattice of topological groups.

Three central themes:

1. **Topologies on a Clifford semigroup**.
   - Disjoint union topology (*strong semilattice of topological groups*) isolates fibers (*discreteness of idempotents*);
   - Compact Hausdorff case (Yeager & Bowman topologies);
   - Metrizability (Bokalo conjecture);
   - Constructive solution (Bowman case).
2. **Hilbert fifth problem for Clifford semigroups**. When all the maximal subgroups are Lie groups?
3. **A rigidity result**. C^1 regularity at idempotents implies strong semilattice of topological groups.

Płonka sums

Semilattice directed system

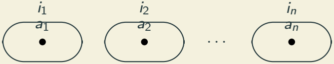
Let Ω be a **plural** algebraic language.

Definition

A **semilattice directed system** of Ω-algebras $\mathbb{A}$ is a functor $\Phi : \mathbf{I} \to \mathsf{Alg}_\Omega$.

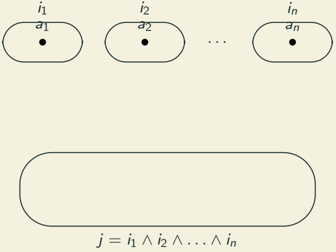
- $\Phi(i) = \mathbf{A}_i$ **fiber**;
- $\Phi(i \leq j) = \varphi_{ji} : \mathbf{A}_j \to \mathbf{A}_i$ **transition homomorphism**.

A **Płonka sum** over such a semilattice directed system is a τ-algebra $\mathbf{A}$ with **universe** $A = \bigsqcup_{i \in I} A_i$ such that the **operations are computed as follows** ($f \in \Omega_n$):



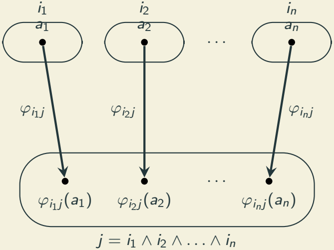
Płonka sums

A **Płonka sum** over such a semilattice directed system is a τ-**algebra A** with **universe** $A = \bigsqcup_{i \in I} A_i$ such that the **operations are computed as follows** ($f \in \Omega_n$):

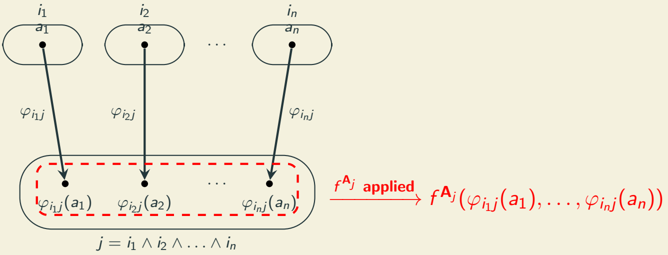


Płonka sums

A **Płonka sum** over such a semilattice directed system is a τ-**algebra A** with **universe** $A = \bigsqcup_{i \in I} A_i$ such that the **operations are computed as follows** $(f \in \Omega_n)$:



A **Płonka sum** over such a semilattice directed system is a τ-**algebra A** with **universe** $A = \bigsqcup_{i \in I} A_i$ such that the **operations are computed as follows** ($f \in \Omega_n$):



Płonka sums

Definition

- An identity $\alpha \approx \beta$ is **regular** if $Var(\alpha) = Var(\beta)$;
- A variety $\mathcal{V}$ is **strongly irregular** if $\mathcal{V} \models p(x, y) \approx x$, where p binary term s.t. $Var(p) = \{x, y\}$;
- $R(\mathcal{V}) \equiv$ variety axiomatized by all the regular identities holding in $\mathcal{V}$.

Theorem (J. Płonka 1969)

$R(\mathcal{V}) \equiv$ class of Płonka sums of algebras in $\mathcal{V}$, if $\mathcal{V}$ is strongly irregular.

An important example: Clifford semigroups

Definition

Inverse semigroup $\equiv$ semigroup S s.t.

$$\forall x \in S, \exists! y \in S : xyx = x \text{ and } yxy = y.$$

Clifford semigroup $\equiv$ inverse semigroup S s.t.

$$S \models xx^{-1} \approx x^{-1}x.$$

Theorem (A. H. Clifford, 1941)

Clifford semigroups $\equiv$ **Płonka sums of groups**.

Topologies on a Płonka sum

Disjoint Union Topology

Let $\mathcal{T}$ be a compatible topology on a Płonka sum $\mathbf{A}$. **TFAE**:

- $\mathcal{T}$ is (isomorphic to) the disjoint union topology;
- $\mathbf{A}_i \in \mathcal{T}$ for every $i \in I$;
- the quotient topology (w.r.t π) on $\mathbf{I}$ is the discrete topology.

The compact hausdorff case

Theorem (D.P Yeager, 1976)

There exists **at most one** topology turning a Clifford semi-group into a compact Hausdorff topological semigroup.

The compact hausdorff case

Theorem (D.P Yeager, 1976)

There exists **at most one** topology turning a Clifford semigroup into a compact Hausdorff topological semigroup.

Theorem (Bonzio, Loi, Z., 2026/2027)

Let $\mathbf{A} \in \mathcal{R}(\mathcal{V})$, where $\mathcal{V}$ is a strongly irregular variety. There exists **at most one** topology turning $\mathbf{A}$ into a compact Hausdorff topological algebra.

This topology is called **Yeager topology**. It is referred to as the **Bowman topology** if $\mathbf{I}$ is a **perfect semilattice** (w.r.t. the quotient topology induced by π).

Metrizability: the Bokalo Conjecture

The Bokalo Conjecture

Conjecture (B. Bokalo)

The Yeager topology of a Clifford semigroup is metrizable iff its semilattice and every fiber are metrizable.

The Bokalo Conjecture

Conjecture (B. Bokalo)

The Yeager topology of a Clifford semigroup is metrizable iff its semilattice and every fiber are metrizable.

Theorem (T. Banakh, 2001)

The Bokalo conjecture is independent of ZFC-axioms:

- **True** under **MA** $+ \neg$**CH**;
- **False** under **CH**.

The Bokalo Conjecture

Theorem (T. Banakh, 2001)

The Bokalo conjecture holds true for the Bowman topology of a Clifford semigroup.

The Bokalo Conjecture

Theorem (T. Banakh, 2001)

The Bokalo conjecture holds true for the Bowman topology of a Clifford semigroup.

Theorem (Bonzio, Loi, Z., 2026/2027)

Let $\mathcal{V}$ be a strongly irregular variety. The Bokalo conjecture holds true for the Bowman topology of a Płonka sum in $R(\mathcal{V})$.

Domain Theory

The Way Below Relation ($\ll$)

Let $(P, \leq)$ be a poset. We say $x \ll y$ (x **way below** y) if for every non-empty up-directed subset $D \subseteq P$ whose supremum exists:

$$y \leq \sup D \implies x \leq d \quad \text{for some } d \in D.$$

Domain Theory: Core Concepts

The Way Below Relation ($\ll$)

Let $(P, \leq)$ be a poset. We say $x \ll y$ (x **way below** y) if for every non-empty up-directed subset $D \subseteq P$ whose supremum exists:

$$y \leq \sup D \implies x \leq d \quad \text{for some } d \in D.$$

DCPOs and Domains

- A poset $(P, \leq)$ is a **DCPO** if every nonempty up-directed subset $D \subseteq P$ has a supremum;
- A DCPO P is a **domain** if

$$x = \sup\{y \in P \mid y \ll x\}, \text{ for every } x \in P.$$

The Lawson Topology & Bases

Definition (Lawson Topology)

Let P be a domain. The **Lawson topology** on P is the topology generated by the sets of the form

$$(\twoheaduparrow x) \setminus (\uparrow F),$$

where $x \in P$, $F \in \mathscr{P}(P)$, and $\twoheaduparrow x = \{y \in P \mid x \ll y\}$.

Definition (Lawson Topology)

Let P be a domain. The **Lawson topology** on P is the topology generated by the sets of the form

$$(\twoheaduparrow x) \setminus (\uparrow F),$$

where $x \in P$, $F \in \mathscr{P}(P)$, and $\twoheaduparrow x = \{y \in P \mid x \ll y\}$.

A subset $\mathcal{B} \subseteq P$ is a **basis** if for every $x \in P$ the set $\mathcal{B}_x = \{b \in \mathcal{B} \mid b \ll x\}$ is up-directed and $x = \sup \mathcal{B}_x$.

Theorem (Gierz et al.). A domain P admits a **countable basis** $\iff$ it is **second countable** w.r.t. the **Lawson topology**.

Lawson and Perfect Semilattices

Definitions

- An Hausdorff topological semilattice S is a **Lawson semilattice** (Lawson, 1969) if S admits a basis consisting of open subsemilattices.
- A **Perfect semilattice** (Bowman, 1971) is a compact Lawson semilattice.

Lawson and Perfect Semilattices

Definitions

- An Hausdorff topological semilattice S is a **Lawson semilattice** (Lawson, 1969) if S admits a basis consisting of open subsemilattices.
- A **Perfect semilattice** (Bowman, 1971) is a compact Lawson semilattice.

Fundamental Theorem of Compact Semilattices (Gierz et al.)

Perfect semilattice $\Rightarrow$ S **Compact Hausdorff domain equipped with the Lawson topology**.

Explicit metric

Explicit metric

Theorem (Bonzio, Loi, Z., 2026/2027)

Let $\mathcal{V}$ be a strongly irregular variety and $\mathbf{A} \in R(\mathcal{V})$ a compact Hausdorff topological Płonka sum with a Lawson metrizable semilattice and metrizable fibers. Then $\mathbf{A}$ is metrizable.

Explicit metric

Theorem (Bonzio, Loi, Z., 2026/2027)

Let $\mathcal{V}$ be a strongly irregular variety and $\mathbf{A} \in R(\mathcal{V})$ a compact Hausdorff topological Płonka sum with a Lawson metrizable semilattice and metrizable fibers. Then $\mathbf{A}$ is metrizable.

1) Fix a **compatible metric** $0 \leq \rho < 1$ on $\mathbf{I}$;

2) $\mathbf{I}$ compact metrizable $\Rightarrow$ $\mathbf{I}$ second countable $\Rightarrow$ $\mathbf{I}$ admits a **countable (domain theoretic) basis** $\mathcal{B} = \{b_j\}_{j \geq 1}$;

3) For each $b \in \mathcal{B}$ fix **a compatible metric** $0 \leq d_b < 1$ on $\mathbf{A}_b$, **a basepoint** $c_b \in A_b$ and a **countable dense subset** $D_b = \{t_{b,k}\}_{k \geq 1} \subseteq A_b$.

Explicit metric

Lemma

For every $i \in I$ and $g \neq h$ in A_i, there exists $b \in \mathcal{B}$ s.t. $b \ll i$ and $\varphi_{i,b}(g) \neq \varphi_{i,b}(h)$.

Proof. It follows by the fact that $\mathcal{B}_i = \{b \in \mathcal{B} \mid b \ll i\}$ is up-directed and such that $\bigvee \mathcal{B}_i = i$, and $\Phi : \mathbf{I} \to \mathrm{Alg}_\Omega$ is ILP.

Explicit metric

Lemma

For every $i \in I$ and $g \neq h$ in A_i, there exists $b \in \mathcal{B}$ s.t. $b \ll i$ and $\varphi_{i,b}(g) \neq \varphi_{i,b}(h)$.

Proof. It follows by the fact that $\mathcal{B}_i = \{b \in \mathcal{B} \mid b \ll i\}$ is up-directed and such that $\bigvee \mathcal{B}_i = i$, and $\Phi : \mathbf{I} \to \mathrm{Alg}_\Omega$ is ILP.

4) Define $a_b : I \to [0,1]$ as $a_b(i) = \rho(i, I \setminus {\uparrow} b)$ for every $b \in \mathcal{B} \setminus \{0\}$, and $a_0 \equiv 1$.

Lemma

For every $i \in I$ and $g \neq h$ in A_i, there exists $b \in \mathcal{B}$ s.t. $b \ll i$ and $\varphi_{i,b}(g) \neq \varphi_{i,b}(h)$.

Proof. It follows by the fact that $\mathcal{B}_i = \{b \in \mathcal{B} \mid b \ll i\}$ is up-directed and such that $\bigvee \mathcal{B}_i = i$, and $\Phi : \mathbf{I} \to \mathrm{Alg}_\Omega$ is ILP.

4) Define $a_b : I \to [0,1]$ as $a_b(i) = \rho(i, I \setminus {\uparrow}b)$ for every $b \in \mathcal{B} \setminus \{0\}$, and $a_0 \equiv 1$.

Remark

$a_b(i) > 0 \iff b \ll i$.

Explicit metric

5) For $i \in I$ and $b \in \mathcal{B}$, **define** the extended transition homomorphism $\widetilde{\varphi}_{i,b} : A_i \to A_b$ such that $\widetilde{\varphi}_{i,b} = \varphi_{i,b}$ if $b \leq i$, $\widetilde{\varphi}_{i,b} \equiv c_b$ otherwise;

6) For $b \in \mathcal{B}$ and $(i, g),\ (j, h) \in A$, **define**

$$P_b((i,g),(j,h)) = |a_b(i) - a_b(j)| +$$

$$+ \sum_{k \geq 1} 2^{-k} |a_b(i) d_b(\widetilde{\varphi}_{i,b}(g), t_{b,k}) - a_b(j) d_b(\widetilde{\varphi}_{i,b}(h), t_{b,k})|;$$

7) Define

$$d((i,g),(j,h)) = \rho(i,j) + \sum_{j \geq 1} 2^{-j} P_{b_j}((i,g),(j,h)).$$

Explicit metric

Theorem (Bonzio, Loi, Z. 2026/2027)

The function d is a metric on A compatible with the topology of $\mathbf{A}$.

We prove identity of indescirnibles.

- Suppose $d((i,g),(j,h)) = 0 \Rightarrow \rho(i,j) = 0 \Rightarrow i = j$;
- Suppose by contradiction $g \neq h$, then there exists $b \in \mathcal{B}$ s.t. $b \ll i$ and $\varphi_{i,b}(g) \neq \varphi_{j,b}(h)$;
- $b \ll i \Rightarrow a_b(i) > 0$, hence $P_b((i,g),(i,h)) = 0$ implies $d_b(\varphi_{i,b}(g), t_{b,k}) = d_b(\varphi_{i,b}(h), t_{b,k})$ for every $k \geq 1$;
- By density of $\{t_{b,k}\}_{k\geq 1}$, we deduce $\varphi_{i,b}(g) = \varphi_{i,b}(h)$, a contradiction.

Explicit metric

Theorem (Bonzio, Loi, Z. 2026/2027)

Let $\mathcal{V}$ be a strongly irregular variety and $\mathbf{A} \in \mathcal{R}(\mathcal{V})$ be a compact Hausdorff topological Płonka sum with a metrizable semilattice, metrizable fibers and isometric (w.r.t to fixed compatible metrics) transition homomorphisms. Then $\mathbf{A}$ is metrizable.

Proof. Fix a compatible metric ρ on $\mathbf{I}$ and a compatible metric d_i for every $\mathbf{A}_i$. Define

$$d((i,g),(j,h)) = \rho(i,j) + d_{i\wedge j}(\varphi_{i,i\wedge j}(g), \varphi_{j,i\wedge j}(h))$$

Then d is a compatible metric on $\mathbf{A}$.

Thank you!

Questions? Remarks?

- S. Bonzio, A. Loi, and G. Zecchini, *Topological and differentiable aspects of Clifford semigroups*, arXiv:2604.23326 [math.GN], 2026.
- S. Bonzio, F. Paoli, and M. Pra Baldi.
  Logics of Variable Inclusion.
  Springer, Trends in Logic, 2022.
- D. P. Yeager, *On the topology of a compact inverse Clifford semigroup*, Trans. Amer. Math. Soc. **215** (1976), 253–267.
- T. T. Bowman, *A Construction Principle and Compact Clifford Semigroups*, Semigroup Forum **2** (1971), 343–353.
- J. D. Lawson, *Topological semilattices with small semilattices*, J. London Math. Soc. (2) 1 (1969) 719–724.
- T. Banakh, *On cardinal invariants and metrizability of topological inverse Clifford semigroups*, Topology and its Applications 128 (1) (2003) 13–48.
- G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, *Continuous Lattices and Domains*, Cambridge Univ. Press, Cambridge, 2003.

Definition

Suppose $\Phi : \mathbf{I} \to \mathsf{Alg}_\Omega$ is an ILP semilattice directed system. Let $\mathcal{T}_{\mathbf{I}}$ and $\mathcal{T}_j$ be compact Hausdorff topologies on $\mathbf{I}$ and $\mathbf{A}_j$ (for every $j \in I$), respectively, and suppose transition homomorphisms are continuous.

For each $U \in \mathcal{T}_{\mathbf{I}},\ j \in U,\ V \in \mathcal{T}_{\mathbf{A}_j}$, we define

$$W(U,(j,V)) = \bigcup_{i \in U \cap (\uparrow j)} (\varphi_{ij})^{-1}(V).$$

Yeager and Bowman topologies

Definition: Yeager's Topology, $\mathcal{T}_Y \subseteq \mathscr{P}(A)$ (D.P. Yeager, 1976)

For every $O \in \mathcal{P}(A)$: $O \in \mathcal{T}_{\mathbf{Y}} \iff$ for every $(i, a) \in O$, there exists $U \in \mathcal{T}_{\mathbf{I}}$ such that:

1. $i \in U$;
2. For every $j \in U \cap {\downarrow} i$, there exists $V \in \mathcal{T}_{\mathbf{A}_j}$ such that:
   (i) $(i, a) \in W(U, (j, V)) \subseteq O$;
   (ii) For every $(z, b) \in W(U, (j, V))$ and $k \in U \cap {\downarrow} z$, there exists $W \in \mathcal{T}_{\mathbf{A}_k}$ such that $(z, b) \in W(U, (k, W)) \subseteq O$.

Definition (T.T. Bowman, 1971)

Bowman topology $\equiv$ Yeager topology for a Płonka sum over a *perfect semilattice*.