

Interpolation in the Inquisitive Hierarchy

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Using their semantics, they showed that the lattice of axiomatic extensions $\mathcal{E}(\mathbf{InqB})$ of **InqB** forms a dual $\omega + 1$ type chain.

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Our goal is to characterize the logics in $\mathcal{E}(\mathbf{InqB})$ that have interpolation!

Interpolation Characterizations

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About the more general question of how interpolation distributes in the lattice of extensions of a logic.

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A logic $\mathbf{L}$ with an implication $\rightarrow$ has Craig interpolation (CIP) if:

If $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$, then $\exists \delta$ s.t. $\text{Var}(\delta) \subseteq \text{Var}(\varphi) \cap \text{Var}(\psi)$, $\vdash_{\mathbf{L}} \varphi \rightarrow \delta$, and $\vdash_{\mathbf{L}} \delta \rightarrow \psi$.

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- Characterizations of interpolation in lattices of extensions have been carried out in many areas of logic:
 - intuitionistic logics (e.g. Maskimova 1977, 2005), modal logics (e.g. Maksimova 2005, Santschi and Vooijs 2026), substructural logics (e.g. Fussner and Santschi 2024, 2024), etc...

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 - Even a few others at TACL 2026: Bi-intuitionistic logics (Almieda and Chen), Bimodal Logics (Bezhanishvili and Chen), Sugihara Algebras (Fussner and Krawczyk)!

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- The distribution of interpolation tends to be *sparse*.

The case of inquisitive logic provides a distinct example: A logic with infinitely many extensions *all of which have interpolation*!

The Main Theorem

Main Theorem

Every logic in the Inquisitive Hierarchy $\mathcal{E}(\mathbf{InqB})$ has Craig interpolation.

- For $\mathbf{InqB}$ the result is known e.g. Fjellstad 2026 or Ciardelli, Iemhoff, and Yang 2019.

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- For the proper extensions the same methods don't readily apply.
 - It is not clear if the proof theory is modular enough to cover these cases.
 - No other logic in $\mathcal{E}(\mathbf{InqB})$ has DP.
- We treat these cases algebraically.
 - We give a modified interpolation via amalgamation style argument.

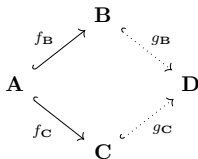
Interpolation via Amalgamation

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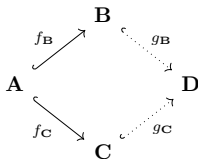
A class of algebras $\mathcal{K}$ has the Amalgamation Property (AP) if each doubly injective span $\langle f_B : \mathbf{A} \rightarrow \mathbf{B}, f_C : \mathbf{A} \rightarrow \mathbf{C} \rangle$ can be completed in $\mathcal{K}$:



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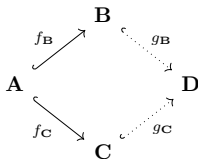


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It is well known that amalgamation in $\mathcal{V}(\mathbf{L})$ is equivalent to CIP for a superintuitionistic logic, $\mathbf{L}$.

The nonstandard semantics of $\mathbf{InqB}$ forces us to revise the standard bridge theorem between AP and CIP.

- The problem: Lindenbaum algebras of logics in $\mathcal{E}(\mathbf{InqB}) \neq$ the free algebras in the corresponding varieties (Yes, inquisitive logics still correspond to varieties).

Algebraic Methods in Non substitution Invariant Contexts

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 - Propositional team logics, dependence logics, etc..
 - Logics of material inference.

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- Non-substitution invariant logics appear more and more often:
 - DEL, PAL, etc...
 - Propositional team logics, dependence logics, etc..
 - Logics of material inference.
- We think the methods here can be generalized to provide insight into many of these examples.
 - The AP/CIP correspondence we present is typical of a more general pattern for adapting bridge theorems.

Inquisitive Logic Algebraically

The variety of Kriesel-Putnam algebras, or KP-algebras, KP is a class of Heyting algebras defined by:

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Inquisitive Varieties

An *inquisitive variety* $\mathcal{K}$ is a class of Heyting algebras obtained by closing a variety of KP-algebras under core super algebras.

core super algebras:

$$\mathbb{C}(\mathcal{K}) = \{\mathbf{B} \mid \exists \mathbf{A} \in \mathcal{K} (\mathbf{A} \leq \mathbf{B} \ \& \ \mathbf{R}(\mathbf{A}) = \mathbf{R}(\mathbf{B}))\}$$

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Each inquisitive variety is also a variety.

For KP, $\mathbb{C}(\text{KP})$ is the variety of ND algebras.

Inquisitive Logic Algebraically

Bezhanishvili et. al. 2021 obtain sound and complete semantics for **InqB** by interpreting formulas in algebras $\mathbf{A}$ in $\mathbb{C}(\text{KP})$ with respect to *regular valuations*.

A *regular valuation* is a valuation $v : \text{Var} \rightarrow A$ where $v(p) \in R(\mathbf{A})$ for $p \in \text{Var}$.

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In particular:

$$\mathbf{A} \in \mathbb{C}(\text{KP}) \text{ if and only if } \mathbf{A} \models_{\mathbf{R}} \mathbf{InqB}$$

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Each logic $\mathbf{L}$ in $\mathcal{E}(\mathbf{InqB})$ corresponds to an inquisitive variety $\mathcal{X}(\mathbf{L})$ in this way.

This establishes a dual lattice isomorphism (Bezhanishvili et. al. 2021).

The Special Role of Regularly Generated Algebras

Within inquisitive varieties *regularly generated*, or simply *regular*, Heyting algebras play a special role.

A Heyting algebra $\mathbf{A}$ is *regularly generated* if it is generated by $\mathbf{R}(\mathbf{A})$.

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Regular Completeness

Let $\mathcal{X}$ be an inquisitive variety. Then $\mathbb{X}(\mathcal{X}_R) = \mathcal{X}$.

$\mathcal{X}_R$ is the class of regularly generated members of $\mathcal{X}$ and $\mathbb{X}(-)$ brings a class to the least inquisitive variety containing it.

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We can now state our transfer theorem!

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All and only the regular members of $\mathcal{X}(\mathbf{L})$ have this property.

The Main Lemma

In light of the transfer theorem it is enough to show:

Lemma

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Proof has two main steps:

- Step 1: show that every span of algebras from $\mathcal{X}_{RFSI}$ has an amalgam in $\mathcal{X}_R$.
- Step 2: show that step 1 implies $\mathcal{X}_R$ has AP.

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Step 2: Showing if that every span of algebras from $\mathcal{X}_{RF SI}$ has an amalgam in $\mathcal{X}_R$, then $\mathcal{X}(\mathbf{L})_R$ has AP.

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Proof: Gratzner's Lemma (Gratzner 1971), subdirect representations, etc.

Must see that every regular is a subdirect product of regular SIs.

Must take a subalgebra of the amalgam obtained by Gratzner's Lemma since an infinite product of regulars may not be regular.

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The key: members of $\mathcal{X}_{RFSI}$ can be characterized as the fixed points an adjunction $\mathbb{C}(\text{KP}) \dashv \text{BA}$.

The operation $R(-) : \mathbb{C}(\text{KP}) \rightarrow \text{BA}$ extends to functor right adjoint to a functor

$$H(-) : \text{BA} \rightarrow \mathbb{C}(\text{KP})$$

that takes a Boolean algebra to its *inquisitive extension* (Bezhanishvili et. al. 2019, 2021).

The Inquisitive Extension

In the inquisitive extension $H(\mathbf{A})$ of a Boolean algebra $\mathbf{A}$ is the freest regular KP-algebra whose regular elements are isomorphic to $\mathbf{A}$.

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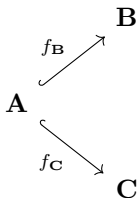
$$\mathbf{A} \cong H(R(\mathbf{A})).$$

Furthermore, for each *proper* inquisitive variety $\mathcal{X}$ there is a *maximal algebra* $\mathbf{A}$ in $\mathcal{X}_{RF SI}$, and thus with the property that $\mathbf{A} \cong H(R(\mathbf{A}))$

Keep this in mind!

Step One Continued

With the intention of showing that every span of algebras from $\mathcal{X}_{RFSI}$ has an amalgam in $\mathcal{X}_R$ take a span of such algebras:



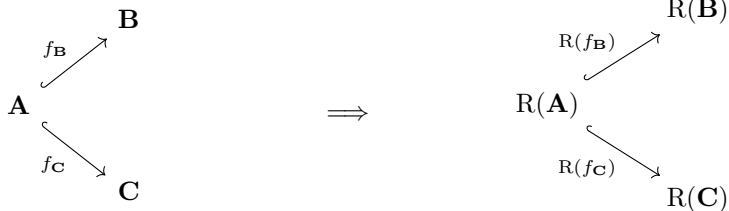
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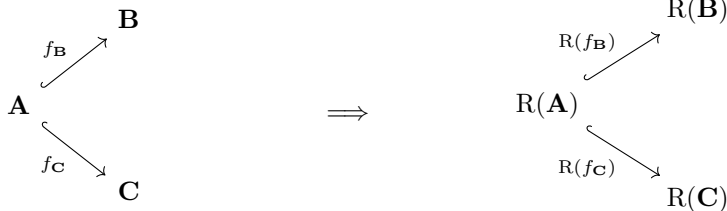
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Do we amalgamate in BA and to transfer back to $\mathbb{C}(KP)$ via $H(-)$?

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NO!

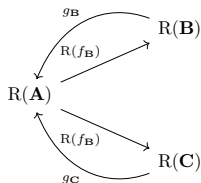
Amalgam obtained in BA may be too large!

Step One Continued

Finite Boolean algebras are absolute retracts in BA!

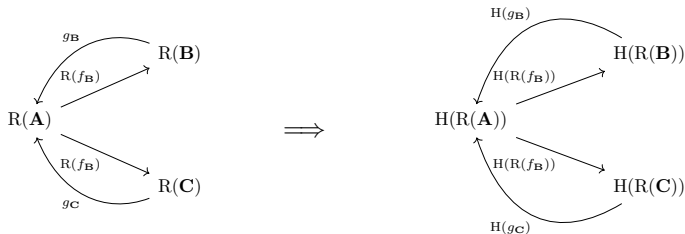
Step One Continued

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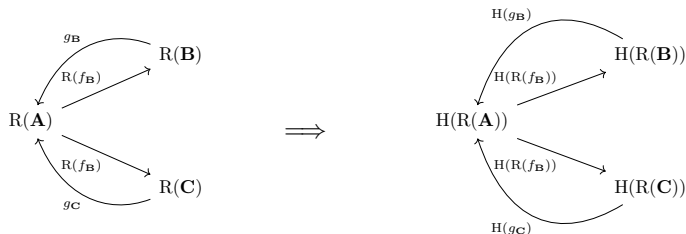
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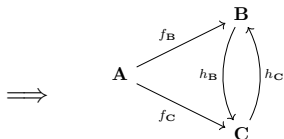


Step One Continued

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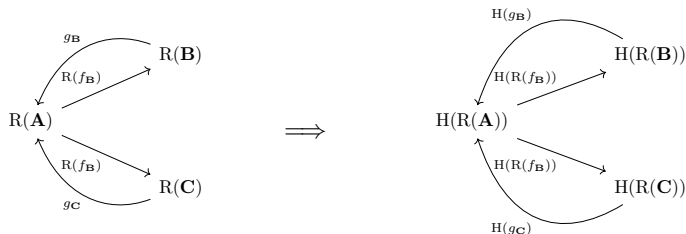


Work through the diagram/unit of adjunction to get a *connected span* in $\mathcal{X}_{RFSI}$.

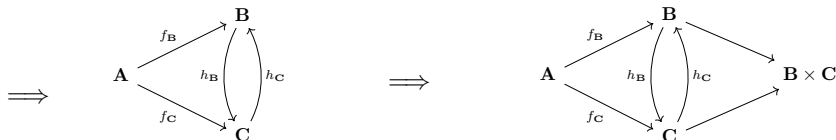


Step One Continued

Finite Boolean algebras are absolute retracts in BA!



Work through the diagram/unit of adjunction to get a *connected span* in $\mathcal{X}_{RFSI}$.



$\mathcal{X}_R$ is closed under finite products, so we are done!

The Main Theorem Again

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 - Leads to further results about **InqB** and extensions e.g. implies each inquisitive logic has Beth definability.
- Suggests general algebraic methods for non-substitution invariant logics.
 - Immediately, these methods apply to other DNA logics (How many DNA logics have CIP?)
 - More generally, the bridge theorem appears to apply to any Weakly Algebraizable logics (Nakov and Quadrellaro 2026).

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 - Immediately, these methods apply to other DNA logics (How many DNA logics have CIP?)
 - More generally, the bridge theorem appears to apply to any Weakly Algebraizable logics (Nakov and Quadrellaro 2026).
- Finally, for the general study of the distribution of interpolation the example is distinct!
 - Seems to relate to failure of substitution invariance.
 - This failure weakens the degree of AP needed in the bridge theorem (only a few of these varieties possess AP).

Thank You :)

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