Interpolation in the Inquisitive Hierarchy

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Using their semantics, they showed that the lattice of axiomatic extensions $\mathcal{E}(\mathbf{InqB})$ of $\mathbf{InqB}$ forms a dual $\omega + 1$ type chain.

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Our goal is to characterize the logics in $\mathcal{E}(\mathbf{InqB})$ that have interpolation!

Interpolation Characterizations

The second thing:

About the more general question of how interpolation distributes in the lattice of extensions of a logic.

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A logic $\mathbf{L}$ with an implication $\to$ has Craig interpolation (CIP) if:

If $\vdash_{\mathbf{L}} \varphi \to \psi$, then $\exists \delta$ s.t. $\mathsf{Var}(\delta) \subseteq \mathsf{Var}(\varphi) \cap \mathsf{Var}(\psi)$, $\vdash_{\mathbf{L}} \varphi \to \delta$, and $\vdash_{\mathbf{L}} \delta \to \psi$.

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- Characterizations of interpolation in lattices of extensions have been carried out in many areas of logic:
  - intuitionistic logics (e.g. Maskimova 1977, 2005), modal logics (e.g. Maksimova 2005, Santschi and Vooijs 2026), substructural logics (e.g. Fussner and Santschi 2024, 2024), etc...

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  - Even a few others at TACL 2026: Bi-intuitionistic logics (Almieda and Chen), Bimodal Logics (Bezhanishvili and Chen), Sugihara Algebras (Fussner and Krawczyk)!

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- The distribution of interpolation tends to be *sparse*.

The case of inquisitive logic provides a distinct example: A logic with infinitely many extensions *all of which have interpolation*!

The Main Theorem

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Every logic in the Inquisitive Hierarchy $\mathcal{E}(\mathbf{InqB})$ has Craig interpolation.

- For $\mathbf{InqB}$ the result is known e.g. Fjellstad 2026 or Ciardelli, Iemhoff, and Yang 2019.

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- For the proper extensions the same methods don't readily apply.
  - It is not clear if the proof theory is modular enough to cover these cases.
  - No other logic in $\mathcal{E}(\mathbf{InqB})$ has DP.
- We treat these cases algebraically.
  - We give a modified interpolation via amalgamation style argument.

Interpolation via Amalgamation

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A class of algebras $\mathcal{K}$ has the Amalgamation Property (AP) if each doubly injective span $\langle f_{\mathbf{B}} : \mathbf{A} \to \mathbf{B}, f_{\mathbf{C}} : \mathbf{A} \to \mathbf{C} \rangle$ can be completed in $\mathcal{K}$:

$$\begin{tikzcd} & \mathbf{B} \arrow[dr, hook, dotted, "g_{\mathbf{B}}"] & \\ \mathbf{A} \arrow[ur, hook, "f_{\mathbf{B}}"] \arrow[dr, hook, "f_{\mathbf{C}}"'] & & \mathbf{D} \\ & \mathbf{C} \arrow[ur, hook, dotted, "g_{\mathbf{C}}"'] & \end{tikzcd}$$

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The nonstandard semantics of $\mathbf{InqB}$ forces us to revise the standard bridge theorem between AP and CIP.

- The problem: Lindenbaum alegbras of logics in $\mathcal{E}(\mathbf{InqB}) \neq$ the free algebras in the corresponding varieties (Yes, inquisitive logics still correspond to varieties).

Algebraic Methods in Non substitution Invariant Contexts

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  - Logics of material inference.

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Algebraic methods for non-substitution invariant logics.

- Non-substitution invariant logics appear more and more often:
  - DEL, PAL, etc...
  - Propositional team logics, dependence logics, etc..
  - Logics of material inference.
- We think the methods here can be generalized to provide insight into many of these examples.
  - The AP/CIP correspondence we present is typical of a more general pattern for adapting bridge theorems.

Inquisitive Logic Algebraically

The variety of Kriesel-Putnam algebras, or KP-algebras, KP is a class of Heyting algebras defined by:

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Inquisitive Varieties

An *inquisitive variety* $\mathcal{X}$ is a class of Heyting algebras obtained by closing a variety of KP-algebras under core super algebras.

core super algebras:

$$\mathbb{C}(\mathcal{K}) = \{\mathbf{B} \mid \exists \mathbf{A} \in \mathcal{K}(\mathbf{A} \leq \mathbf{B} \ \&\ \mathrm{R}(\mathbf{A}) = \mathrm{R}(\mathbf{B}))\}$$

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Each inquisitive variety is also a variety.

For KP, $\mathbb{C}(\mathsf{KP})$ is the variety of ND algebras.

Inquisitive Logic Algebraically

Bezhanishvili et. al. 2021 obtain sound and complete semantics for $\mathbf{InqB}$ by interpreting formulas in algebras $\mathbf{A}$ in $\mathbb{C}(\mathsf{KP})$ with respect to *regular valuations*.

A *regular valuation* is a valuation $v : \mathsf{Var} \to A$ where $v(p) \in \mathrm{R}(\mathbf{A})$ for $p \in \mathsf{Var}$.

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In particular:

$$\mathbf{A} \in \mathbb{C}(\mathsf{KP}) \text{ if and only if } \mathbf{A} \vDash_{\mathrm{R}} \mathbf{InqB}$$

Where $\vDash_{\mathrm{R}}$ is validity with respect to the class of regular valuations.

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Each logic $\mathbf{L}$ in $\mathcal{E}(\mathbf{InqB})$ corresponds to an inquisitive variety $\mathcal{X}(\mathbf{L})$ in this way.

This establishes a dual lattice isomorphism (Bezhanishvili et. al. 2021).

The Special Role of Regularly Generated Algebras

Within inquisitive varieties *regularly generated*, or simply *regular*, Heyting algebras play a special role.

A Heyting algebra $\mathbf{A}$ is *regularly generated* if it is generated by $\mathrm{R}(\mathbf{A})$.

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Regular Completeness

Let $\mathcal{X}$ be an inquisitive variety. Then $\mathbb{X}(\mathcal{X}_R) = \mathcal{X}$.

$\mathcal{X}_R$ is the class of regularly generated members of $\mathcal{X}$ and $\mathbb{X}(-)$ brings a class to the least inquisitive variety containing it.

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Finite Subdirectly Irreducible Regular Completeness

Let $\mathcal{X}$ be an inquisitive variety. Then $\mathbb{X}(\mathcal{X}_{RFSI}) = \mathcal{X}$.

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We can now state our transfer theorem!

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Theorem

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For inquisitive logics: the Lindenbaum Algebras of $\mathbf{L} \neq$ the free algebras in $\mathcal{X}(\mathbf{L})$.

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All and only the regular members of $\mathcal{X}(\mathbf{L})$ have this property.

The Main Lemma

In light of the transfer theorem it is enough to show:

Lemma

If $\mathcal{X}$ be a proper inquisitive variety, then $\mathcal{X}_R$ has AP.

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Proof has two main steps:

- Step 1: show that every span of algebras from $\mathcal{X}_{RFSI}$ has an amalgam in $\mathcal{X}_R$.
- Step 2: show that step 1 implies $\mathcal{X}_R$ has AP.

Starting with Step Two

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Proof: Gratzer's Lemma (Gratzer 1971), subdirect representations, etc.

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Proof: Gratzer's Lemma (Gratzer 1971), subdirect representations, etc.

Must see that every regular is a subdirect product of regular SIs.

Must take a subalgebra of the amalgam obtained by Gratzer's Lemma since an infinite product of regulars may not be regular.

Step One

How do we show that every span of algebras from $\mathcal{X}_{RFSI}$ has an amalgam in $\mathcal{X}_R$?

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The operation $\mathrm{R}(-) : \mathbb{C}(\mathsf{KP}) \to \mathsf{BA}$ extends to functor right adjoint to a functor

$$\mathrm{H}(-) : \mathsf{BA} \to \mathbb{C}(\mathsf{KP})$$

that takes a Boolean algebra to its *inquisitive extension* (Bezhanishvili et. al. 2019, 2021).

The Inquisitive Extension

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Crucially, for each member $\mathbf{A}$ of $\mathcal{X}_{RFSI}$ we have:

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Furthermore, for each *proper* inquisitive variety $\mathcal{X}$ there is a *maximal algebra* $\mathbf{A}$ in $\mathcal{X}_{RFSI}$, and thus with the property that $\mathbf{A} \cong \mathrm{H}(\mathrm{R}(\mathbf{A}))$

Keep this in mind!

Step One Continued

With the intention of showing that every span of algebras from $\mathcal{X}_{RFSI}$ has an amalgam in $\mathcal{X}_R$ take a span of such algebras:

$$\begin{tikzcd} & \mathbf{B} \\ \mathbf{A} \arrow[ur, hook, "f_{\mathbf{B}}"] \arrow[dr, hook, "f_{\mathbf{C}}"'] & \\ & \mathbf{C} \end{tikzcd}$$

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Do we amalgamate in BA and to transfer back to $\mathbb{C}(\mathsf{KP})$ via $\mathrm{H}(-)$?

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$$

Do we amalgamate in BA and to transfer back to $\mathbb{C}(\mathsf{KP})$ via $\mathrm{H}(-)$?

NO!

Amalgam obtained in BA may be too large!

Step One Continued

Finite Boolean algebras are absolute retracts in BA!

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\mathbf{A} \arrow[ur, "f_{\mathbf{B}}"] \arrow[dr, "f_{\mathbf{C}}"'] & & \mathbf{B} \times \mathbf{C} \\
 & \mathbf{C} \arrow[u, bend right, "h_{\mathbf{C}}"'] \arrow[ur] &
\end{tikzcd}
$$

$\mathcal{X}_R$ is closed under finite products, so we are done!

The Main Theorem Again

Theorem

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- Suggests general algebraic methods for non-substitution invariant logics.
  - Immediately, these methods apply to other DNA logics (How many DNA logics have CIP?)
  - More generally, the bridge theorem appears to apply to any Weakly Algebraizble logics (Nakov and Quadrellaro 2026).

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  - Immediately, these methods apply to other DNA logics (How many DNA logics have CIP?)
  - More generally, the bridge theorem appears to apply to any Weakly Algebraizble logics (Nakov and Quadrellaro 2026).
- Finally, for the general study of the distribution of interpolation the example is distinct!
  - Seems to relate to failure of substitution invariance.
  - This failure weakens the degree of AP needed in the bridge theorem (only a few of these varieties posses AP).

Thank You :)

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