Duality for quasi-Nelson algebras with recovery operators

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- Goal of this talk: a Priestley-style duality for quasi-Nelson algebras enriched with recovery operators.

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  Quasi-Nelson algebras may be represented as twist-algebras over two Heyting algebras, related by back-and-forth maps: this allows us to combine seamlessly the three approaches above.

Quasi-Nelson algebras

Figure 22—A Half-Nelson. One of Its Many Applications

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A quasi-Nelson algebra (QNA) is a commutative, integral, bounded residuated lattice $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$ satisfying the Nelson equation:

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- Idempotent members ($x * x = x$) are precisely Heyting algebras.

Quasi-Nelson algebras & recovery operators

The two basic operators $\circ, \bullet$ are defined on a QNA by:

$$x \vee {\sim} x \vee y = 1 \qquad \text{if and only if} \qquad \bullet x \leq y$$

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both operators are unique if they exist, and both are implicitly defined on every complete QNA.

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Thus, we have four independent operators expressing:

consistency $\circ$, inconsistency ${\sim}\circ$, determinedness ${\sim}\bullet$ and undeterminedness $\bullet$.

The twist representation

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Given Heyting algebras $\mathrm{H}_+, \mathrm{H}_-$, maps $n\colon H_+ \to H_-$ (bounded lattice hom.), $p\colon H_- \to H_+$ (meet- and bound-preserving), with $n \circ p = id_{H_-}$, $id_{H_+} \leq p \circ n$, and a dense filter $\nabla \subseteq H_+$, define the twist-algebra as follows:

$$\mathrm{Tw}\langle \mathrm{H}_+, \mathrm{H}_-, n, p, \nabla\rangle := \{\langle a_+, a_-\rangle : a_+ \vee p(a_-) \in \nabla,\ a_+ \wedge p(a_-) = 0_+\}$$

$$\begin{aligned}
\langle a_+, a_-\rangle \wedge \langle b_+, b_-\rangle &:= \langle a_+ \wedge_+ b_+, a_- \vee_- b_-\rangle \\
\langle a_+, a_-\rangle \vee \langle b_+, b_-\rangle &:= \langle a_+ \vee_+ b_+, a_- \wedge_- b_-\rangle \\
\langle a_+, a_-\rangle * \langle b_+, b_-\rangle &:= \langle a_+ \wedge_+ b_+, (n(a_+) \to_- b_-) \wedge_- (n(b_+) \to_- a_-)\rangle. \\
\langle a_+, a_-\rangle \Rightarrow \langle b_+, b_-\rangle &:= \langle (a_+ \to_+ b_+) \wedge_+ (p(b_-) \to_- p(a_-)), n(a_+) \wedge_- b_-\rangle \\
0 := \langle 0_+, 1_-\rangle \qquad 1 &:= \langle 1_+, 0_-\rangle.
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Representation (Rivieccio–Spinks)

Every quasi-Nelson algebra is isomorphic to some $\mathrm{Tw}\langle \mathrm{H}_+, \mathrm{H}_-, n, p, \nabla \rangle$.

Recovery operators on twist-algebras

The dual pseudo-complement $\neg$ on a bounded lattice is the unique operation s.t.:

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$$\bullet\langle a_+, a_- \rangle = \langle \ulcorner_+ (a_+ \vee_+ p(a_-)), \neg_- n(\ulcorner_+ (a_+ \vee_+ p(a_-))) \rangle$$
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Since $\bullet x = \bullet(x \vee \sim x)$ and $\circ x = \circ(x \wedge \sim x)$, the above can be simplified as:

$$\bullet\langle a_+, 0_- \rangle = \langle \urcorner_+ a_+, \neg_- n(\urcorner_+ a_+) \rangle$$
$$\circ\langle 0, a_- \rangle = \langle \neg_+ p(\urcorner_- a_-), \urcorner_- a_- \rangle.$$

where $a_+ \in \nabla$ and $a_- \in n[\nabla]$.

Recovery operators on twist-algebras

Representation (Figallo & Rivieccio)

Each $\langle \mathrm{A}, \bullet \rangle$ (resp. $\langle \mathrm{A}, \circ \rangle$) is isomorphic to a twist-algebra $Tw\langle \mathrm{H}_{+}, \mathrm{H}_{-}, n, p, \nabla \rangle$ s.t. every element of ∇ (resp. $n[\nabla]$) has a dual pseudo-complement.

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Lemma

Let H be a Heyting algebra (or pseudo-complemented distributive lattice) such that every dense element has a dual pseudo-complement in H. Since, for all $a \in H$,

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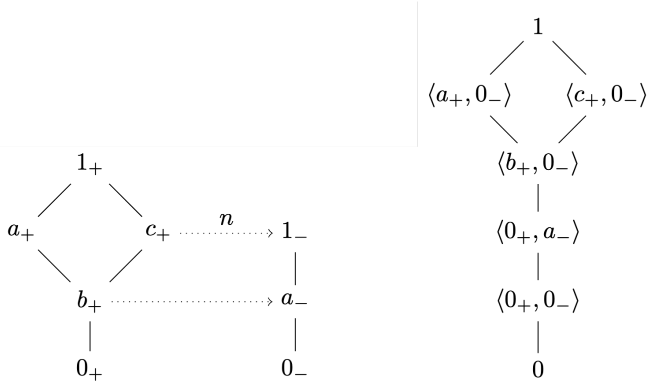
Representation (Jansana & Rivieccio)

Each $\langle \mathrm{A}, \bullet \rangle$ (resp. $\langle \mathrm{A}, \circ \rangle$) is isomorphic to a twist-algebra $Tw\langle \mathrm{H}_+, \mathrm{H}_-, n, p, \nabla \rangle$ where H_+ (resp. H_-) is dually pseudo-complemented.

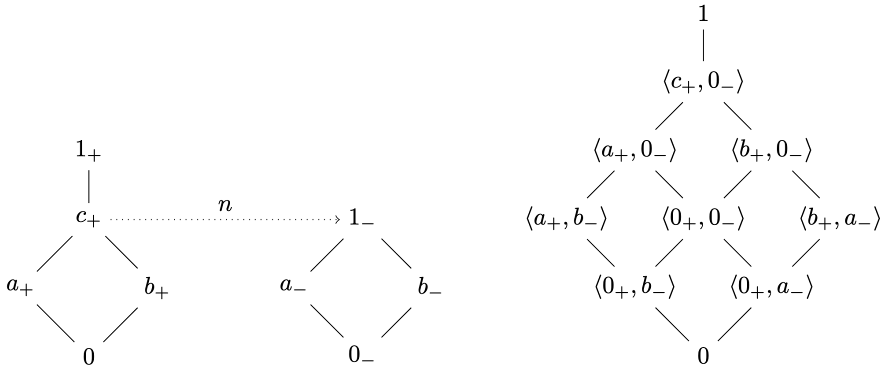
Example 1



$\mathbf{H}_+$ $\quad$ $\mathbf{H}_-$ $\quad$ $\mathbf{Tw}\langle \mathbf{H}_+, \mathbf{H}_-, n, p, H_+ \rangle$

Removing $\langle 0_+, 0_- \rangle$ we have the other possible twist-algebra: either disproves $\circ x \leq \sim \bullet x$.

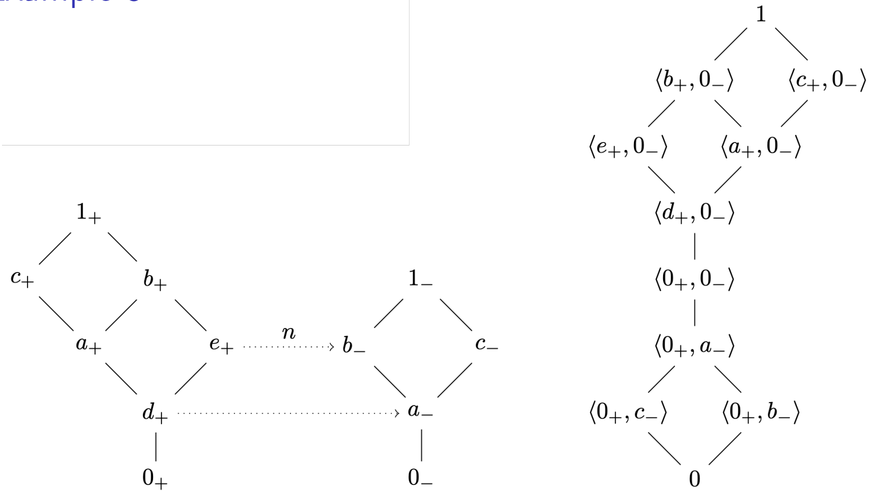
Example 2



$\mathbf{H}_+$ $\mathbf{H}_-$ $\mathbf{Tw}\langle \mathbf{H}_+, \mathbf{H}_-, n, p, H_+ \rangle$

We have two non-isomorphic sub-twist-algebras (for $\nabla := \uparrow a_+$ $\nabla := \uparrow c_+$).

Example 3



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- Esakia duality: restriction to Heyting algebras $\simeq^{op}$ Esakia spaces (down-sets of opens are open).

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- Priestley (Esakia) duality restricts to a duality between:
  - dually pseudo-complemented distributive lattices (Heyting algebras);
  - Priestley (Esakia) spaces satisfying (dps)
    (Spatial morphisms are required to satisfy (dpf)).

Two-sorted duality for QNA

Key observation

A tuple $\langle \mathrm{H}_+, \mathrm{H}_-, n, p, \nabla \rangle$, consisting of Heyting algebras with meet-preserving maps, may be viewed as a diagram in the framework of Bezhanishvili-Jansana duality for distributive meet-semilattices.

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Notation

Given a relation $R \subseteq X \times Y$ (between Priestley spaces), $X' \subseteq X$ and $Y' \subseteq Y$, let:

$$R[X'] := \{y \in Y : \exists x \in X' \text{ s.t. } \langle x, y \rangle \in R\}$$

$$\Box_R Y' := \{x \in X : R[x] \subseteq Y'\}.$$

Two-sorted duality for QNA

A quasi-Nelson space is a tuple $X = \langle X_+, X_-, R_n, R_p, \mathcal{C} \rangle$ such that:

1. X_+, X_- are Esakia spaces.
2. The relations $R_n \subseteq X_- \times X_+$, $R_p \subseteq X_+ \times X_-$ satisfy:
   (2.1) $R_n[x_-]$ and $R_p[x_+]$ are non-empty closed up-sets, for all $x_-$and x_+;
   (2.2) for all $U_- \in \mathsf{L}(X_-)$, $U_+ \in \mathsf{L}(X_+)$, $\Box_{R_p} U_- \in \mathsf{L}(X_+)$ and $\Box_{R_n} U_+ \in \mathsf{L}(X_-)$.
3. R_n is functional, i.e., for all $x_- \in X_-$ there is $x_+ \in X_+$ such that ${\uparrow} x_+ = R_n[x_-]$.
4. R_p is an Esakia relation, i.e., for all $x_+ \in X_+$, $x_- \in X_-$ s.t. $\langle x_+, x_- \rangle \in R_p$, there is $y_+ \in {\uparrow} x_+$ such that $R_p[y_+] = {\uparrow} x_-$.
5. $\mathcal{C} \subseteq X_+$ is a closed set such that $\mathcal{C} \subseteq \max(X_+)$.
6. $\leq_{X_-} = R_p \circ R_n$.
7. $R_n \circ R_p \subseteq \leq_{X_+}$.

Two-sorted duality for QNA

Proposition 1

From a tuple $\langle \mathsf{H}_+, \mathsf{H}_-, n, p, \nabla \rangle$ corresponding to a QNA, we obtain a quasi-Nelson space

$$\langle X(\mathsf{H}_+), X(\mathsf{H}_-), X(n), X(p), \mathcal{C}_\nabla \rangle$$

letting:

$$X(n) := \{\langle x_-, x_+ \rangle \in X(\mathsf{L}_-) \times X(\mathsf{L}_+) : n^{-1}[x_-] \subseteq x_+\}$$

$$X(p) := \{\langle x_+, x_- \rangle \in X(\mathsf{L}_+) \times X(\mathsf{L}_-) : p^{-1}[x_+] \subseteq x_-\}$$

$$\mathcal{C}_\nabla := \bigcap \{\phi(a) : a \in \nabla\}, \ \textit{where } \phi(a) = \{x \in X(\mathsf{H}_+) : a \in x_+\}.$$

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Proposition 2

Conversely, a quasi-Nelson space $X = \langle X_+, X_-, R_n, R_p, \mathcal{C} \rangle$ gives us a quasi-Nelson tuple

$$\langle L(X_+), L(X_-), \Box_{R_n}, \Box_{R_p}, \nabla_\mathcal{C} \rangle$$

where

$$\nabla_\mathcal{C} := \{U \in L(X_+) : \mathcal{C} \subseteq U\}.$$

Two-sorted duality for QNA

Morphisms (spatial)

A morphism between QN-spaces spaces $X = \langle X_+, X_-, R_n, R_p, \mathcal{C} \rangle$ and $X' = \langle X'_+, X'_-, R'_n, R'_p, \mathcal{C}' \rangle$ consists of a pair of Esakia functions $f = \langle f_+, f_- \rangle$ with $f_+ \colon X_+ \to X'_+$ and $f_- \colon X_- \to X'_-$ s.t.:

1. f preserves R_p and R_n, that is, if $\langle x_+, x_- \rangle \in R_p$, then $\langle f_+(x_+), f_-(x_-) \rangle \in R'_p$, etc.
2. f_+ and f_- are bounded morphisms, in the sense that
   1. if $\langle f_+(x_+), x'_- \rangle \in R'_p$, then there is $x_- \in X_-$ such that $f_-(x_-) \leq'_- x'_-$ and $\langle x_+, x_- \rangle \in R_p$.
   2. if $\langle f_-(x_-), x'_+ \rangle \in R'_n$, then there is $x_+ \in X_+$ such that $f_+(x_+) \leq'_+ x'_+$ and $\langle x_-, x_+ \rangle \in R_n$.
3. $f_+[\mathcal{C}] \subseteq \mathcal{C}'$.

Two-sorted duality for QNA

Morphisms (algebraic)

A morphism between quasi-Nelson tuples $\langle \mathrm{H}_+, \mathrm{H}_-, n, p, \nabla \rangle$ and $\langle \mathrm{H}'_+, \mathrm{H}'_-, n', p', \nabla'_+ \rangle$ consists of a pair of Heyting algebra homomorphisms $h = \langle h_+, h_- \rangle$ with $h_+ \colon H_+ \to H'_+$ and $h_- \colon H_- \to H'_-$ such that $h_+[\nabla] \subseteq \nabla'_+$, $h_+ \circ p = p' \circ h_-$ and $n' \circ h_+ = h_- \circ n$.

$$\begin{tikzcd}
\mathrm{H}_+ \arrow[r, shift left, "n"] \arrow[d, "h_+"] & \mathrm{H}_- \arrow[l, shift left, "p"] \arrow[d, "h_-"] \\
\mathrm{H}'_+ \arrow[r, shift left, "n'"] & \mathrm{H}'_- \arrow[l, shift left, "p'"]
\end{tikzcd}$$

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The functors X and L are defined on morphisms pair-wise as per Priestley duality.

Two-sorted duality for QNA

Theorem (Duality)

The functors X and L establish a dual equivalence of categories between quasi-Nelson tuples and quasi-Nelson spaces.

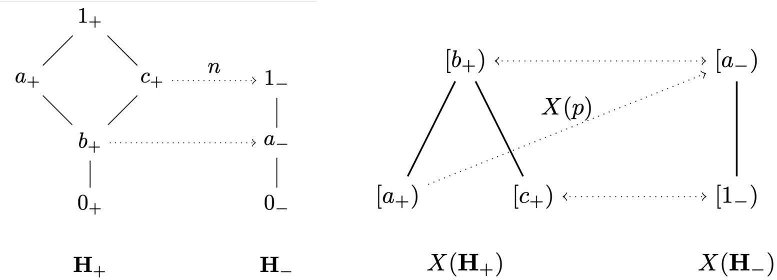
Duality with recovery operators

Duality with recovery operators

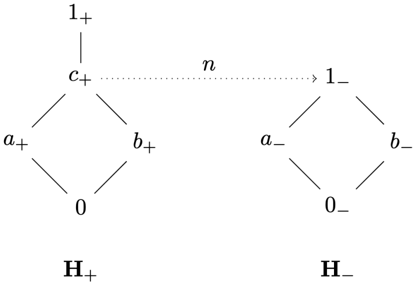
A duality for quasi-Nelson tuples with recovery operator(s) can be obtained by:

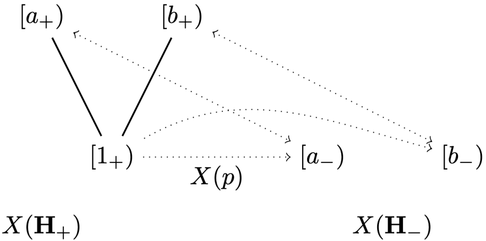
- Restricting algebraic objects to tuples $\langle \mathrm{H}'_+, \mathrm{H}'_-, n', p', \nabla'_+ \rangle$ such that H'_+ and/or H'_- are dually pseudo-complemented.
- Considering (pairs of) Esakia spaces that satisfy (dps), i.e. such that the up-set of every clopen down-set is open.
- Restricting algebraic morphisms to (pairs of) Heyting homomorphisms that also preserve the dual pseudo-complement.
- Restricting spatial morphisms to (pairs of) Esakia functions that satisfy (dpf).

Example 1, dually

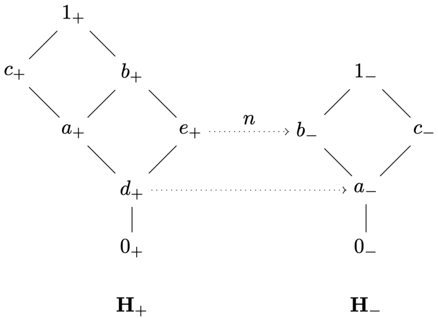


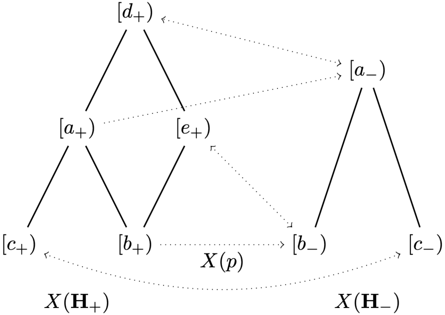
Example 2, dually





Example 3, dually





Sample dual characterizations

Algebraic properties of the recovery operators translate into structural properties on dual space $X = \langle X_+, X_-, R_n, R_p, \mathcal{C} \rangle$.

Letting $\mathsf{Tw}(X) := \langle L(X_+), L(X_-), \Box_{R_n}, \Box_{R_p}, \nabla_{\mathcal{C}} \rangle$, we have, for example:

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- $\mathrm{Tw}(X) \vDash {\sim} \bullet x = \circ x$ iff R_n preserves minimal points along ${\downarrow} x_-$ (property (dpf)).

Future work

- A one-sorted duality for QN algebras with recovery operators, not relying on the twist representation – extending R. Cignoli's duality for Nelson algebras and S. Celani's duality for distributive lattices with negation.

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- Dual study of notable subvarieties: further axioms on $\circ, \bullet$ or specializing the underlying algebra (e.g. prelinear QN-algebras, twist-products over Gödel algebras).
- Alternative recovery operators considered in the literature, such as those where $\bullet a$ ($\circ a$) is always a Boolean (or an explosive) element. These are not yet representable in twist form – a natural next challenge for the duality.

Thanks!


That's all Folks!


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