Sequent Calculi for Semiring Semantics

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Background

- Application of techniques in substructural logics to semiring semantics.
- Substructural logics: well studied group of non-classical logic; algebraically the logics of residuated lattices.
- Semiring semantics: introduced as framework for annotated databases.
- Substantive literature on model theory. Relatively little work on proof theory.

Background

- Application of techniques in substructural logics to semiring semantics.
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Selection of Model Theory Investigations

- Status of Ehrenfeucht-Fraïssé and other model checking games [1].
- Versions of (finite) Łos-Tarski preservation theorem [2]
- Recovery of Hanf locality for general semirings and Gaifman locality for min-max semirings [3].

Background

- By contrast, extensive literature on proof theory for substructural logics - sequent calculi.
- Semirings and residuated lattices are algebraically related.
- Broad idea; exploit this relationship to find sequent calculi for semiring semantics.
- Fairly early stage of investigation; have proof systems for some well behaved cases.

Semiring Semantics

Definition

A (commutative naturally ordered) *semiring* is an algebra $(S, +, \cdot, 0, 1)$ such that:

- $(S, +, 0)$ and $(S, \cdot, 1)$ are commutative monoids;
- $\cdot$ distributes over $+$;
- $0 \neq 1$ and $0 \cdot s = s \cdot 0 = 0$;
- The order $s \leq t$ iff $\exists r : s + r = t$ is a partial order.

A semiring is:

- *idempotent* iff $+$ is idempotent;
- *multiplicativly idempotent* iff $\cdot$ is idempotent;
- *absorptive* iff $s + st = s$ for all $s, t \in S$, equivalently iff 1 is the top element for the natural order.

Semiring Semantics

Example

- *Fuzzy semiring* $\mathbb{F} = ([0,1], max, min, 0, 1)$,
- *Viterbi semiring* $\mathbb{V} = ([0,1], max, \cdot, 0, 1)$,
- *Łukasiewicz semiring* $\mathbb{L} = ([0,1], max, \odot, 0, 1)$,
- *Bag semiring* $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$,
- *Provenance semirings*: quotients of the semiring of polynomials $\mathbb{N}[X]$, particularly the free absorptive semiring $\mathbb{S}[X]$.

Semiring Semantics

Definition

For a semiring S, a (finite) domain M and a (relational) predicate language τ, a mapping $\pi\colon Lit(\tau, A) \to S$ is an *S-interpretation*. We say it is *model-defining* iff for each literal pair $R(\bar{m})_i, \neg R(\bar{m}) \in Lit$ exactly one of $\pi[R(\bar{m})], \pi[\neg R(\bar{m})]$ is 0.
We extend an S-interpretation to a evaluation of formulas φ in *negation normal form* by:

$$\pi(\varphi \vee \psi) := \pi(\varphi) +^s \pi(\psi) \;\; \pi(\varphi \wedge \psi) := \pi(\varphi) \cdot^s \pi(\psi).$$

$$\pi(\exists x \varphi(x)) := \sum_{a \in S} \pi(\varphi(a)) \;\; \pi(\forall x \varphi(x)) := \prod_{a \in S} \pi(\varphi(a)).$$

Semiring Semantics

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Given a semiring S we define the propositional/predicate logic of S by:

$$\Gamma \models \Delta \text{ iff for all } S\text{-interpretations } \pi \; \prod_{\gamma \in \Gamma} \pi(\gamma) \leq \sum_{\delta \in \Delta} \pi(\delta).$$

Relationship with Residuated Lattices

Lemma

label=i. *The $\{\vee, \cdot\}$-reduct of a (commutative) residuated lattice is a semiring.*

label=ii. *If S is a semiring whose natural order is a lattice order and join-complete then S has a residuated lattice expansion.*

Example

- *Fuzzy semiring* $\mathbb{F} \mapsto [0,1]_{\mathbb{G}}$
- *Viterbi semiring* $\mathbb{V} \mapsto [0,1]_{\Pi}$,
- *Łukasiewicz semiring* $\mathbb{L} \mapsto [0,1]_{\mathbb{L}}$,
- *Bounded Bag semiring* $\mathbb{N} \cup \{\omega\} \mapsto (\mathbb{N} \cup \{\omega\}, \to)$,
- The free absorptive semiring $\mathbb{S}[X] \mapsto \mathbb{S}[X]^{\to} \in FL_{ew}$. The $(\vee, \cdot)$-reduct of any $A \in FL_{ew}$ is a absorptive semiring.

Propositional Łukasiewicz

Definition

Let π be a $[0,1]$-interpretation. For a finite multiset of formulas Γ we define

$$*^{\pi}(\Gamma) = 1 - \Sigma_{\gamma \in \Gamma}(1 - \pi(\gamma)).$$

A hypersequent $H = \Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$ is valid iff for all interpretations π $\exists 1 \leq i \leq n$:

$$*^{\pi}(\Gamma_i) \leq *^{\pi}(\Delta_i).$$

Definition

Given an absorptive semiring/residuated lattice S and a hypersequent calculus $\mathcal{C}$ we say that:

- $\mathcal{C}$ is S-sound iff every derivable hypersequent in $\mathcal{C}$ is S-valid, i.e. $\vdash_{\mathcal{C}} H$ implies $\models_S H$.
- $\mathcal{C}$ is S-complete iff every S-valid hypersequent is derivable in $\mathcal{C}$, i.e. $\models_S H$ implies $\vdash_{\mathcal{C}} H$.

$$\frac{}{H \mid \varphi \Rightarrow \varphi}\ (\mathrm{id}) \qquad \frac{}{H \mid \Rightarrow}\ (\mathrm{emp}) \qquad \frac{}{H \mid \Gamma, 0 \Rightarrow \varphi}\ (0 \Rightarrow_{\mathbb{L}})$$

$$\frac{H \mid}{H \mid G \mid}\ (\mathrm{EW}) \qquad \frac{H \mid G \mid G \mid}{H \mid G \mid}\ (\mathrm{EC}) \qquad \frac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma, \Pi \Rightarrow \Delta}\ (\mathrm{W})$$

$$\frac{H \mid \Gamma, \Pi \Rightarrow \Delta, \Sigma}{H \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma}\ (\mathrm{Split}) \qquad \frac{H \mid \Gamma \Rightarrow \Delta \qquad H \mid \Pi \Rightarrow \Sigma}{H \mid \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\mathrm{Mix})$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \gamma}\ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ (\Rightarrow \vee_1)$$

$$\frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ (\Rightarrow \vee_2)$$

$$\frac{H \mid \Gamma, \varphi, \psi \Rightarrow \Delta \qquad H \mid \Gamma, 0 \Rightarrow \Delta}{H \mid \Gamma, \varphi \cdot \psi \Rightarrow \Delta}\ (\cdot \Rightarrow_{\mathbb{L}})$$

$$\frac{H \mid \Gamma \Rightarrow \varphi, \psi, \Delta \mid \Gamma \Rightarrow 0, \Delta}{H \mid \Gamma \Rightarrow \varphi \cdot \psi, \Delta}\ (\Rightarrow \cdot_{\mathbb{L}})$$

Propositional Łukasiewicz

Theorem

The calculus $\mathbb{LC}$ *is sound and complete for* $\mathbb{L}$.

Propositional Łukasiewicz

Theorem

The calculus $\mathbb{LC}$ is sound and complete for $\mathbb{L}$.

Proof Sketch.

- Each logical rule is invertible - reduces problem to showing all valid literal hypersequents are derivable.
- Devise new rules for primitive negation that read upwards aim to eliminate conflicting literal pairs.
- Derivability of valid hypersequents without conflicting literals is the same as for Ł.
- Whilst these rules are not invertible in general, can determine conditions under which they are. □

$$\frac{H \mid \Gamma, p \Rightarrow 0, \Delta \qquad H \mid \Gamma, 0 \Rightarrow \neg p, \Delta}{H \mid \Gamma, p \Rightarrow \neg p, \Delta}\ (+\Rightarrow\neg)$$

$$\frac{H \mid \Gamma, \neg p \Rightarrow 0, \Delta \qquad H \mid \Gamma, 0 \Rightarrow p, \Delta}{H.\Gamma, \neg p, \Rightarrow p, \Delta}\ (\neg\Rightarrow+)$$

$$\frac{H \mid \Gamma, 0 \Rightarrow \Delta}{H \mid \Gamma, p, \neg p \Rightarrow \Delta}\ (+, \neg \Rightarrow) \qquad \frac{H \mid \Gamma \Rightarrow 0, 0, \Delta}{H \mid \Gamma \Rightarrow p, \neg p, \Delta}\ (\Rightarrow+, \neg)$$

$$\frac{H \mid \Gamma, 0 \Rightarrow \Delta \mid \Pi, \neg p \Rightarrow \Sigma \qquad H \mid \Gamma, p \Rightarrow \Delta \mid \Pi, 0 \Rightarrow \Sigma}{H \mid \Gamma, p \Rightarrow \Delta \mid \Pi, \neg p \Rightarrow \Sigma}\ (+|\neg\Rightarrow)$$

$$\frac{H \mid \Gamma \Rightarrow 0, \Delta \mid \Pi \Rightarrow \neg p, \Sigma \qquad H \mid \Gamma \Rightarrow p, \Delta \mid \Pi \Rightarrow 0, \Sigma}{H \mid \Gamma \Rightarrow p, \Delta \mid \Pi \Rightarrow \neg p, \Sigma}\ (\Rightarrow+|\neg)$$

$$\frac{H \mid \Gamma, 0 \Rightarrow, \Delta \mid \Pi \Rightarrow \neg p, \Sigma \qquad H \mid \Gamma, p \Rightarrow \Delta \mid \Pi \Rightarrow 0, \Sigma}{H \mid \Gamma, p \Rightarrow \Delta \mid \Pi \Rightarrow \neg p, \Sigma}\ (+|\Rightarrow|\neg)$$

$$\frac{H \mid \Gamma \Rightarrow 0, \Delta \mid \Pi, \neg p \Rightarrow \Sigma \qquad H \mid \Gamma \Rightarrow p, \Delta \mid \Pi, 0 \Rightarrow \Sigma}{H \mid \Gamma \Rightarrow p, \Delta \mid \Pi, \neg\ p \Rightarrow \Sigma}\ (\neg|\Rightarrow|+)$$

Propositional Łukasiewicz

Lemma

Each of the following invertibility properties hold:

- If $\neg p \notin \Gamma$ then $(+\Rightarrow\neg)$ is invertible and if $p \notin \Gamma$ then $(\neg\Rightarrow+)$ is invertible.
- If $p, \neg p \notin \Delta$ then $(+, \neg\Rightarrow)$ is invertible and if $p, \neg p \notin \Gamma$ then $(\Rightarrow+, \neg)$ are invertible.
- If $\neg p \notin \Gamma \cup \Delta$ and $p \notin \Pi \cup \Sigma$ then $(+|\neg\Rightarrow)$, $(\Rightarrow+|\neg)$, $(+|\Rightarrow|\neg)$ and $(\neg|\Rightarrow|+)$ are all invertible.

Difficulties of First Order

- Universal quantifier is not interpreted as infimum of natural order outside of lattice semirings; size of domain matters for value of quantified formula.
- Breaks comparison with predicate substructural logics; need to elsewhere for inspiration of rules for $\forall$.

$$\frac{\Gamma, \varphi(t) \Rightarrow \Delta}{\Gamma, \forall x(\varphi(x)) \Rightarrow \Delta}\,(\forall \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi(a), \Delta}{\Gamma \Rightarrow \forall x(\varphi(x)), \Delta}\,(\Rightarrow \forall)$$

$$\frac{\Gamma, \varphi(a) \Rightarrow \Delta}{\exists x(\varphi(x)) \Rightarrow \Delta}\,(\exists \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi(t), \Delta}{\exists x(\varphi(x)), \Delta}\,(\Rightarrow \exists)$$

Lattice Semirings

Proof Strategy

- Proceed via a Hilbert system - familiar Henkin-method.
- Ideal is to have an equivalent sequent calculus.
- Successful for calculus with implication
- Reduce form follows from Cut-elimination - hopefully resolves soon.

Lattice Semirings

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- Reduce form follows from Cut-elimination - hopefully resolves soon.

Theorem (Standard completeness - Lattice Semirings)

For all sets of τ-sentences $\Gamma \cup \Delta$ the following are equivalent:

i. $\Gamma \models_L \Delta$ *for all lattice semirings* $L \ncong \mathbb{B}$.

ii. $\Gamma \models_L \Delta$ *for some lattice semiring* $L \ncong \mathbb{B}$.

iii. $\pi[\Delta] = 1$ *for all* S_3*-interpretations* M *with* $\pi[\Gamma] = 1$.

Hilbert System

$$(\mathrm{r})\ \varphi \to \varphi \ (\mathrm{t})\ (\varphi \to \psi) \to ((\psi \to \lambda) \to (\varphi \to \lambda))$$
$$(\mathrm{p})\ (\varphi \to (\psi \to \lambda)) \to (\psi \to (\varphi \to \lambda))$$
$$(\mathrm{u1})\ 1 \quad (\mathrm{u2})\ \varphi \to (1 \to \varphi)$$
$$(\wedge 1)\ (\varphi \wedge \psi) \to \varphi \quad (\wedge 2)\ (\varphi \wedge \psi) \to \psi$$
$$(\wedge 3)\ (\varphi \to \psi \wedge \varphi \to \lambda) \to (\varphi \to (\psi \wedge \lambda))$$
$$(\vee 1)\ \varphi \to (\varphi \vee \psi) \quad (\vee 2)\ \psi \to (\varphi \vee \psi)$$
$$(\vee 3)\ (\varphi \to \lambda \wedge \psi \to \lambda) \to ((\varphi \vee \psi) \to \lambda)$$
$$(\mathrm{w})\ (\varphi \to 1) \wedge (0 \to \varphi)$$
$$(\mathrm{dis})\ (\varphi \wedge (\psi \vee \lambda)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \lambda))\ (\mathrm{prl})\ (\varphi \to \psi) \vee (\psi \to \varphi)$$
$$(\neg 1)\ (R(\bar{t}) \to 0) \vee (\neg R(\bar{t}) \to 0) \quad (\neg 2)\ (R(\bar{t}) \to 0) \wedge \neg R(\bar{t}) \to 0) \to 0$$
$$(\forall 1)\ \forall x \varphi(x) \to \varphi(t) \quad (t \text{ substitutable for } x \text{ in } \varphi)$$
$$(\forall 2)\ \forall x(\varphi \to \psi) \to (\varphi \to \forall x \psi) \quad (x \text{ not free in } \varphi)$$
$$(\forall 3)\ \forall x(\varphi \vee \psi) \to (\varphi \vee \forall x \psi) \quad (x \text{ not free in } \varphi)$$
$$(\exists 1)\ \varphi(t) \to \exists x \varphi(x) \quad (t \text{ substitutable for } x \text{ in } \varphi)$$
$$(\exists 2)\ \forall x(\varphi \to \psi) \to (\exists x \varphi \to \psi) \quad (x \text{ not free in } \psi)$$

Hilbert System

$$\text{(r) } \varphi \to \varphi \text{ (t) } (\varphi \to \psi) \to ((\psi \to \lambda) \to (\varphi \to \lambda))$$

$$\text{(p) } (\varphi \to (\psi \to \lambda)) \to (\psi \to (\varphi \to \lambda))$$

$$\text{(u1) } 1 \quad \text{(u2) } \varphi \to (1 \to \varphi)$$

$$(\wedge 1)\ (\varphi \wedge \psi) \to \varphi \quad (\wedge 2)\ (\varphi \wedge \psi) \to \psi$$

$$(\wedge 3)\ (\varphi \to \psi \wedge \varphi \to \lambda) \to (\varphi \to (\psi \wedge \lambda))$$

$$(\vee 1)\ \varphi \to (\varphi \vee \psi) \quad (\vee 2)\ \psi \to (\varphi \vee \psi)$$

$$(\vee 3)\ (\varphi \to \lambda \wedge \psi \to \lambda) \to ((\varphi \vee \psi) \to \lambda)$$

$$\text{(w) } (\varphi \to 1) \wedge (0 \to \varphi)$$

$$\text{(dis) } (\varphi \wedge (\psi \vee \lambda)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \lambda)) \text{ (prl) } (\varphi \to \psi) \vee (\psi \to \varphi)$$

$$(\neg 1)\ (R(\bar{t}) \to 0) \vee (\neg R(\bar{t}) \to 0) \quad (\neg 2)\ (R(\bar{t}) \to 0) \wedge \neg R(\bar{t}) \to 0) \to 0$$

$$(\forall 1)\ \forall x \varphi(x) \to \varphi(t) \quad (t \text{ substitutable for } x \text{ in } \varphi)$$

$$(\forall 2)\ \forall x(\varphi \to \psi) \to (\varphi \to \forall x \psi) \quad (x \text{ not free in } \varphi)$$

$$(\forall 3)\ \forall x(\varphi \vee \psi) \to (\varphi \vee \forall x \psi) \quad (x \text{ not free in } \varphi)$$

$$(\exists 1)\ \varphi(t) \to \exists x \varphi(x) \quad (t \text{ substitutable for } x \text{ in } \varphi)$$

$$(\exists 2)\ \forall x(\varphi \to \psi) \to (\exists x \varphi \to \psi) \quad (x \text{ not free in } \psi)$$

Hilbert System

Definition

A τ-theory Γ is:

- model-defining iff for all τ-atomic sentences $R(\bar{t})$ exactly one of $\Gamma \vdash R(\bar{t}) \to 0$ or $\Gamma \vdash \neg R(\bar{t}) \to 0$ holds.
- Henkin if for each τ-formula $\varphi(x)$ with one free variable x whenever $\Gamma \not\vdash \forall x \varphi(x)$ then $\Gamma \not\vdash \varphi(c)$ for some τ-constant c.

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- Henkin if for each τ-formula $\varphi(x)$ with one free variable x whenever $\Gamma \nvdash \forall x \varphi(x)$ then $\Gamma \nvdash \varphi(c)$ for some τ-constant c.

Lemma

Let Γ be τ-theory an theory and ψ a sentence. if $\Gamma \nvdash \psi$ then $\Gamma' \nvdash \psi$ for some τ'-theory Γ' which is model-defining and Henkin with $\Gamma \subseteq \Gamma'$ and $\tau \subseteq \tau'$.

Hilbert System

Proof Sketch.

List all atomic τ'-sentences α_i and all τ'-formulas with one free-variable β_i.
If $n = 2i$ we define

$$\psi_{n+1} = \psi_n \text{ and } \Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\alpha_i \to 0\} & \text{if } \Gamma_n \cup \{\alpha_i \to 0\} \nvdash \psi_n \\ \Gamma_n \cup \{\neg\alpha_i \to 0\} & \text{if } \Gamma_n \cup \{\neg\alpha_i \to 0\} \nvdash \psi_n. \end{cases}$$

If $n = 2i + 1$ we let c_i be a fresh τ'-constant not occurring in Γ_n, ψ_n or β_i and define:

$$\psi_{n+1} = \begin{cases} \psi_n \vee \beta_i(c_i) & \text{if } \Gamma_n \nvdash \psi_n \vee \beta_i(c_i) \\ \psi_n & \text{otherwise.} \end{cases}$$

$$\Gamma_{n+1} = \begin{cases} \Gamma_n & \text{if } \Gamma_n \nvdash \psi_n \vee \beta_i(c_i) \\ \Gamma_n \cup \{\psi_n \to \forall x \beta_i(x)\} & \text{otherwise} \end{cases}$$

$\square$

Hilbert System

Definition

Let Γ be a Henkin, model-defining τ-theory and π_Γ be the interpretation into LT_Γ whose domain is the set of variable-free τ-terms (i.e. constant symbols) and $\pi(R(\bar{c})) = [R(\bar{c})]$.

Lemma

Let Γ be a Henkin, model-defining τ-theory. If $\varphi(x)$ is a τ-formula with one-free variable x:

a) $[\forall x \varphi(x)] = inf\{[\varphi(c)] : c \text{ is a } \tau\text{-constant}\}$.

b) $[\exists x \varphi(x)] = sup\{[\varphi(c) : c \text{ is a } \tau\text{-constant}\}$.

If δ is a τ-sentence then:

c) $\pi_\Gamma(\delta) = [\delta]_\Gamma$.

d) $\Gamma \vdash \delta$ *iff* $[1]_\Gamma \leq [\delta]_\Gamma$.

Finally:

e) π_Γ *is a model-defining LT_Γ-interpretation.*

Sequent Calculus

$$\frac{}{\Gamma, \varphi \Rightarrow \varphi, \Delta}\,(\mathrm{id}) \qquad \frac{}{\Gamma, R(\bar{t}), \neg R(\bar{t}) \Rightarrow \Delta}\,(\neg \Rightarrow) \qquad \frac{}{\Gamma, 0 \Rightarrow \Delta}\,(0 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,(\mathrm{Cut}) \quad \frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\,(\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\,(\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\,(\Rightarrow \vee_2)$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\,(\wedge_1 \Rightarrow) \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\,(\wedge_2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}\,(\Rightarrow \wedge)$$

$$\frac{\Gamma, \varphi(t) \Rightarrow \Delta}{\Gamma, \forall x(\varphi(x)) \Rightarrow \Delta}\,(\forall \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi(a), \Delta}{\Gamma \Rightarrow \forall x(\varphi(x)), \Delta}\,(\Rightarrow \forall)$$

$$\frac{\Gamma, \varphi(a) \Rightarrow \Delta}{\exists x(\varphi(x)) \Rightarrow \Delta}\,(\exists \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi(t), \Delta}{\exists x(\varphi(x)), \Delta}\,(\Rightarrow \exists)$$

Sequent Calculus

Additional Rules:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\,(\to\Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\,(\Rightarrow\to)$$

$$\frac{\Gamma, R(\bar{t}) \to 0 \Rightarrow \neg R(\bar{t}) \to 0, \Delta}{\Gamma \Rightarrow \neg R(\bar{t}) \to 0, \Delta}\,(\neg\to)$$

$$\frac{\Gamma, R(\bar{t}) \to 0 \Rightarrow \neg R(\bar{t}) \to 0, \Delta}{\Gamma, R(\bar{t}) \to 0) \Rightarrow \Delta}\,(+\to)$$

Sequent Calculus

Lemma

The Hilbert system and sequent calculus are equivalent.

Proof Sketch.

$$\dfrac{\dfrac{R(\bar{t}) \to 0 \Rightarrow R(\bar{t}) \to 0, \neg R(\bar{t}) \to 0}{\Rightarrow R(\bar{t}) \to 0, \neg R(\bar{t} \to 0)}\,(\neg\to)}{\Rightarrow (R(\bar{t}) \to 0 \vee \neg R(\bar{t}) \to 0)}\,(C),(\vee \Rightarrow)$$

$$\dfrac{\dfrac{\dfrac{R(\bar{t}) \to 0, \neg R(\bar{t}) \to 0 \Rightarrow \neg R(\bar{t}) \to 0, 0}{R(\bar{t}) \to 0, \neg R(\bar{t}) \to 0 \Rightarrow 0}\,(+\to)}{R(\bar{t}) \to 0) \wedge \neg R(\bar{t}) \to 0 \Rightarrow 0}\,(C),(\wedge \Rightarrow)}{\Rightarrow (R(\bar{t}) \to 0 \wedge \neg R(\bar{t}) \to 0) \to 0}\,(\Rightarrow\to)$$

□

Extending Further

Many further questions - broadly two main camps:

Extending to further semirings

- Systems for predicate logics of other absorptive semirings
- Any possibility for non-absorptive semirings.

Making use of what we have

- Investigating usual consequences and further properties of the established proof systems.
- Comparing proof systems; relationship between algebraic property of semirings and resulting proof systems

Thanks

Thank you all for listening.

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