Double Equivalential Algebras

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Double Heyting algebra

Definition

An algebra $\mathbf{A} = (A, \vee, \wedge, \rightarrow, \leftarrow, 0, 1)$ is called a **double Heyting algebra** if $(A, \vee, \wedge, \rightarrow, 0, 1)$ is a Heyting algebra and $(A, \vee, \wedge, \leftarrow, 0, 1)$ is a dual Heyting algebra.
More precisely, $\mathbf{A}$ is a double Heyting algebra if $(A, \vee, \wedge, 0, 1)$ is a bounded lattice and the operations $\rightarrow$ and $\leftarrow$ satisfy the following equivalences:

$$x \wedge y \leq z \iff y \leq x \rightarrow z,$$

$$x \vee y \geq z \iff y \geq z \leftarrow x.$$

Equivalential algebra

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By an **equivalential algebra** we mean an algebra $(A, \leftrightarrow)$ that is a $(\leftrightarrow)$-subreduct of a Heyting algebra, where the operation $\leftrightarrow$ is defined by:

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Axioms (Kabziński and Wroński, 1975)

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Definition

An algebra $(A, \cdot, 0)$ with one binary operation $\cdot$ and one constant 0 is called an **equivalential algebra with zero** if $(A, \cdot)$ is an equivalential algebra and $\mathbf{A}$ satisfies the identity:

$$x00yy = x00.$$

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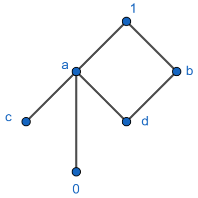
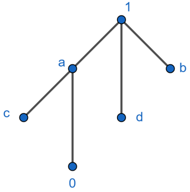
$\mathcal{DE}$ is not congruence orderable.

Orderings



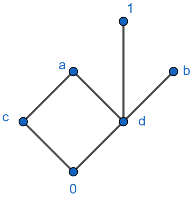
$\leq_e$

$x \leq_e y \iff y \in [x)_e$



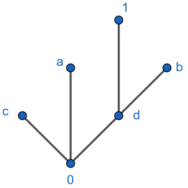
$\ll$

$x \ll y \iff x \cdot y = x$



$\leq_d$

$x \leq_d y \iff x \in (y]_d$



$\ll_d$

$x \ll_d y \iff x \circ y = y$

Failure of local finiteness

Theorem

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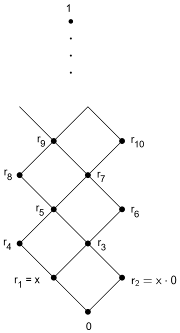

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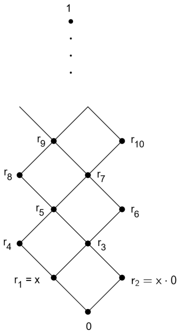

$R = \{0, 1\} \cup \{r_k : k \in \mathbb{N}\}$, where $r_1 = x$, $r_2 = x \cdot 0$, $r_{2n+1} = r_{2n} \circ r_{2n-1}$, $r_{2n+2} = r_{2n+1} \cdot r_{2n-1}$ for $n = 1, 2, \ldots$.

Some basic properties

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Theorem

Let $\mathbf{A} \in \mathcal{DE}$ *and let* $a, b \in A$. *Then:*

1. $(a \circ 1)aa = a \circ 1, \quad a0 \circ a \circ a = a0,$
2. $a(a \circ 1)(a \circ 1) = a, \quad a \circ (a0) \circ (a0) = a,$
3. $(a0 \circ a) \cdot 0 = 0, \quad ((a \circ 1) \cdot a) \circ 1 = 1,$
4. $(a0 \circ a) \cdot a = a00, \quad ((a \circ 1) \cdot a) \circ a = a \circ 1 \circ 1,$
5. $(a0) \cdot (a \circ 1) = a \cdot d(a), \quad (a0) \circ (a \circ 1) = a \circ g(a),$
6. *if* $a << b$, *then* $b0 = 0$ *and* $g(b) = 1$.

Congruences

Theorem

Let $\mathbf{A} \in \mathcal{DE}$*, and let* $\varphi \in \operatorname{Con} \mathbf{A}$*. Then:*

1. $x \in 1/\varphi$ *iff* $x \circ 1 \in 0/\varphi$,
2. $xy \in 1/\varphi$ *iff* $x \circ y \in 0/\varphi$ *iff* $(x \circ y) \cdot 0 \in 1/\varphi$,
3. $x \in 1/\varphi$ *iff* $d(x) \in 1/\varphi$,
4. $0/\varphi = \{x \in A : x0 \in 1/\varphi\}$,
5. $0/\varphi = \{x \circ y : x, y \in 1/\varphi\}$.

Linear double equivalential algebra

Let $n \in \mathbb{N}$. We denote by $\mathbf{n}$ the $(\cdot, \circ)$-reduct of an n-element double Heyting algebra whose underlying set forms a chain.

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Remark

If $\mathbf{A} \in V(\mathbf{n})$, then the following identities hold:

1. $(x \circ y) \cdot 0 = (x \cdot y) \circ 1 \cdot 0,$
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Theorem

Let $\mathbf{A} \in V(\mathbf{n})$, and let $\alpha \in \mathrm{Con}\,\mathbf{A}^e$. Denote $F := 1/\alpha$. Then $\alpha \in \mathrm{Con}\,\mathbf{A}$ if and only if the following conditions are satisfied:

1. *$a \in F$ iff $d(a) \in F$,*
2. *if $a0 \in F$, then $(a \circ z) \cdot z \in F$ for all $z \in A$.*

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Remark

The algebras $\mathbf{3}$, $\mathbf{2}$, and $\mathbf{2}_{0=1}$ are simple.

Foster-Pixley Theorem

Theorem

(Foster-Pixley) Let $\mathbf{S}_1, \ldots, \mathbf{S}_n$ be simple algebras in a congruence-permutable variety V. If

$$C \leq \mathbf{S}_1 \times \cdots \times \mathbf{S}_n$$

is a subdirect product, then

$$C \cong \mathbf{S}_{i_1} \times \cdots \times \mathbf{S}_{i_k}$$

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Theorem

Let $\mathbf{A} \in \mathrm{SP}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$ *with* $1 < |A| < \infty$*. Then* $\mathbf{A} \in \mathrm{IP}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$.

Subdirectly irreducible algebras

Theorem

Let $\mathbf{A} \in \mathrm{HSP}(\mathbf{3})$ *with* $1 < |A| < \infty$. *Then* $\mathbf{A} \in \mathrm{IP}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$.

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Corollary

The only subdirectly irreducible algebras in $V(\mathbf{3})$ are $\mathbf{3}$, $\mathbf{2}$, and $\mathbf{2}_{0=1}$.

The main theorem

Theorem

The variety $V(\mathbf{3})$ is:

1. *semisimple,*
2. *directly representable (i.e., it is finitely generated and has only finitely many finite directly indecomposable members).*

Moreover, if $\mathbf{A} \in V(\mathbf{3})$ and $|A| < \infty$, then $\mathbf{A}$ is congruence uniform (i.e., for every $\varphi \in \operatorname{Con} \mathbf{A}$ and all $a, b \in A$, we have $|a/\varphi| = |b/\varphi|$).

Free algebra

Example

Let $\mathcal{F}(\mathbf{n})$ be the n-generated free algebra in the variety $V(\mathbf{3})$. Then $\mathcal{F}(\mathbf{1})$ is the $(\cdot, \circ)$-reduct of the double Heyting algebra $\mathbf{3} \times \mathbf{2}^2$, with generator x.

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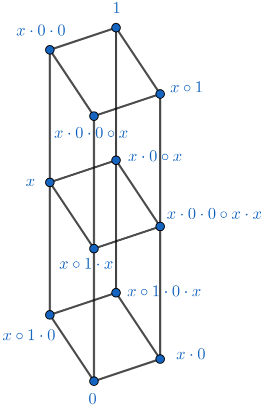

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