

Coherent and ideal actions in ideally exact categories with an application to varieties of universal algebras

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Group action

An **action** of a group G on a group X is a morphism $\phi: G \rightarrow \text{Aut}(X)$ in Grp .

- ▷ The category of the actions of a group G is equivalent to the category of the split epis over G .
- ▷ The equivalence is established via semidirect products.

In 2005, Borceux, Janelidze and Kelly extended the correspondance between **actions** and **split extensions** from the setting of groups to more general *semi-abelian categories*.

Definition

A finitely complete category $\mathcal{C}$ is a *category with semidirect products* if, for all $p : E \rightarrow B$, the functor

$$p^* : \text{Pt}(B) \rightarrow \text{Pt}(E)$$

has a left adjoint and is monadic.

Equivalence between **split epis** over B and **algebras on split epis** over E .

A commutative diagram illustrating the equivalence between split epis over B and algebras on split epis over E . The diagram consists of the following elements:

- Top-left: $(X, \xi) \in$
- Top-right: $\text{Pt}(E)^{T_p}$
- Bottom-left: $(X, \xi) \rtimes (B, p) \in$
- Bottom-middle: $\text{Pt}(B)$
- Bottom-right: $\text{Pt}(E)$
- Arrows:
 - A dashed arrow from $(X, \xi) \in$ to $(X, \xi) \rtimes (B, p) \in$.
 - A solid arrow from $(X, \xi) \in$ to $\text{Pt}(E)^{T_p}$.
 - A solid arrow from $(X, \xi) \rtimes (B, p) \in$ to $\text{Pt}(B)$.
 - A solid arrow from $\text{Pt}(B)$ to $\text{Pt}(E)$ labeled p^* .
 - A solid arrow from $\text{Pt}(E)^{T_p}$ to $\text{Pt}(E)$.
 - A curved arrow from $\text{Pt}(B)$ to $\text{Pt}(E)^{T_p}$ labeled K .
 - A curved arrow from $\text{Pt}(B)$ to $\text{Pt}(E)$ labeled $\perp$.

If $\mathcal{C}$ pointed: choose $p : 0 \rightarrow B$, $\text{Pt}(0) \cong \mathcal{C}$ and equivalence between **split epis** over B and **algebras on their kernels** in $\mathcal{C}$.

A commutative diagram illustrating the equivalence between split epis over B and algebras on their kernels in $\mathcal{C}$. The diagram consists of the following elements:

- Top-left: $(X, \xi) \in$
- Top-right: $\mathcal{C}^{B_b}$
- Bottom-left: $(X, \xi) \rtimes B \in$
- Bottom-middle: $\text{Pt}(B)$
- Bottom-right: $\mathcal{C}$
- Arrows:
 - A dashed arrow from $(X, \xi) \in$ to $(X, \xi) \rtimes B \in$.
 - A solid arrow from $(X, \xi) \in$ to $\mathcal{C}^{B_b}$.
 - A solid arrow from $(X, \xi) \rtimes B \in$ to $\text{Pt}(B)$.
 - A solid arrow from $\text{Pt}(B)$ to $\mathcal{C}$ labeled $\ker$.
 - A solid arrow from $\mathcal{C}^{B_b}$ to $\mathcal{C}$.
 - A curved arrow from $\text{Pt}(B)$ to $\mathcal{C}^{B_b}$ labeled K .
 - A curved arrow from $\text{Pt}(B)$ to $\mathcal{C}$ labeled $\perp$.

where $B_b X = \ker(B + X \xrightarrow{[1,0]} B)$

The B_b -algebras are called **internal actions**.

Case: non-pointed categories

Problem: Non-pointed categories *don't* have kernels, but we can see them in a bigger category.

A **basic setting** for relative U -ideals is an adjunction $U \xrightleftharpoons[U]{F} V$, where V is homological and U is conservative and faithful.

$k: X \rightarrow U(A)$ in V is a **relative U -ideal** of an object A in U if

$$\exists \begin{array}{ccc} A & & X \xrightarrow{k} U(A) \\ \downarrow f & \text{s.t.} & \downarrow \\ B & & 0 \longrightarrow U(B) \end{array} \quad \begin{array}{c} \downarrow U(f) \\ \text{is a pullback.} \end{array}$$

- **The varietal case:** $V = (V, \Sigma_V, Z_V)$, $U = (U, \Sigma_U, Z_U)$ algebraic varieties, V homological (and hence semi-abelian), $\Sigma_V \subseteq \Sigma_U$, $Z_V \subseteq Z_U$ and $U: U \rightarrow V$ forgetful functor.

Definition

A category $\mathcal{U}$ is **ideally exact** if it is Barr exact, protomodular, has finite coproducts and $\iota: 0 \rightarrow 1$ is a regular epi.

Theorem (G. Janelidze)

$\mathcal{U}$ is ideally exact iff it is Barr exact, has finite coproducts and $\exists U: \mathcal{U} \rightarrow \mathcal{V}$ monadic, with $\mathcal{V}$ semi-abelian.

- If $\mathcal{U}$ ideally exact, the pullback functor $\mathcal{U} \rightarrow (\mathcal{U} \downarrow 0)$ along $0 \rightarrow 1$ is monadic

- A monadic adjunction $\mathcal{U} \xrightleftharpoons[U]{F} \mathcal{V}$, where $\mathcal{U}$ ideally exact and $\mathcal{V}$ semi-abelian, is associated to $0 \rightarrow 1$ iff the unit of the adjunction is cartesian.
- A non-trivial algebraic variety is ideally exact iff it is protomodular.

- ▷ **Ideally exact context:** a monadic adjunction with cartesian unit $\mathcal{U} \xrightleftharpoons[U]{F} \mathcal{V}$, with $\mathcal{U}$ ideally exact and $\mathcal{V}$ semi-abelian.

Wajsberg hoops

A **hoop** is an algebra $(A, \cdot, \rightarrow, 1)$ s.t.

- $x \rightarrow x = 1$
- $x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x)$
- $x \cdot y \rightarrow z = x \rightarrow (y \rightarrow z)$

A **Wajsberg hoop** is a hoop s.t.

- $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

A **bounded hoop** is a hoop with a constant 0 s.t.

- $0 \rightarrow x = 1$

We denote by Hoops, WHoops and BWHoops the corresponding categories.

Hoops and WHoops are semi-abelian, while BWHoops is protomodular (and hence ideally exact).

- ▷ Every hoop is endowed with a partial order $\leq$ s.t. $x \leq y$ iff $x \rightarrow y = 1$.
- ▷ Bounded Wajsberg hoops are term equivalent to *MV-algebras*.
- ▷ In BWHoops, initial object: $L_2 = \{0, 1\}$, terminal object: trivial algebra $\{1\}$.

Consider the adjunction $\mathbf{BWHoops} \xrightleftharpoons[U]{M} \mathbf{WHoops}$, where

- U is the functor that forgets the constant 0
- M is the *MV-closure*, i.e. it maps every Wajsberg hoop W to the bounded Wajsberg hoop

$$M(W) = (W \times \{0, 1\}, \cdot, \rightarrow, 0, 1),$$

where $0 := (1, 0)$, $1 := (1, 1)$,

$$(a, i) \cdot (b, j) = \begin{cases} (a \cdot b, 1), & \text{if } i = j = 1, \\ (a \rightarrow b, 0), & \text{if } i = 1, j = 0, \\ (b \rightarrow a, 0), & \text{if } i = 0, j = 1, \\ ((a \rightarrow (a \cdot b)) \rightarrow b, 0), & \text{if } i = j = 0. \end{cases}$$

$$(a, i) \rightarrow (b, j) = \begin{cases} (a \rightarrow b, 1), & \text{if } i = j = 1, \\ (a \cdot b, 0), & \text{if } i = 1, j = 0, \\ ((a \rightarrow (a \cdot b)) \rightarrow b, 1), & \text{if } i = 0, j = 1, \\ (b \rightarrow a, 1), & \text{if } i = j = 0. \end{cases}$$

- ▷ The unit $\eta : 1_{\mathbf{WHoops}} \Rightarrow UM$ of the adjunction is cartesian, and hence the adjunction is associated with $L_2 \rightarrow \{1\}$
- ▷ It gives an ideally exact context

Definition

Let B be an object of U .

- A split epi $A \xrightleftharpoons[p]{p} U(B)$ in V is **ideal** if there exist a split epi $A' \xrightleftharpoons[p']{p'} B$ in U , and an isomorphism $\sigma: U(A') \rightarrow A$ that induces an isomorphism of points.
- A morphism $h: A_1 \rightarrow A_2$ of split epis over $U(B)$ is ideal if there exists h' s.t. $h \circ \sigma_1 = \sigma_2 \circ U(h')$.

Definition

Let $\xi: U(B) \triangleright X \rightarrow X$ be the action associated with the split epi $A \xrightleftharpoons[p]{p} U(B)$, and

$\xi_0: UF(0) \triangleright X \rightarrow X$ the action associated with the canonical $UF(X) \xrightleftharpoons[UF(\iota)]{UF(\tau)} UF(0)$. We say that

- ξ is a **coherent action** if $\xi \circ (U(\iota) \triangleright \text{id}_X) = \xi_0$
- ξ is an **ideal action** if the corresponding split epi is ideal.

Lemma

ξ coherent iff $\exists f: UF(X) \rightarrow A$ in $\mathcal{V}$ such that the following diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{\eta_X} & UF(X) & \begin{array}{c} \xrightarrow{UF(\tau)} \\ \xleftarrow{UF(\iota)} \end{array} & UF(0) \\
 \parallel & & \downarrow f & & \downarrow U(\iota) \\
 X & \xrightarrow{k} & A & \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} & U(B).
 \end{array}$$

is a morphism of split extensions in $\mathcal{V}$. Moreover, the square on the right is a split pullback.

Theorem (M. Mancini, G. Metere, F. P.)

In an ideally exact context, all ideal actions are coherent.

Definition

An ideally exact context admits a **good theory of actions** (it is **BAT**) if all coherent actions are ideal, and all morphisms of such actions are ideal.

Ideally exact context: a varietal example

- U, V : non-trivial varieties, signatures Σ_U, Σ_V , identities Z_U, Z_V
- V semi-abelian, unique constant 0 ; $\Sigma_V \subseteq \Sigma_U, Z_V \subseteq Z_U$
- $\Sigma_U \setminus \Sigma_V$: nullary operations, interpreted as constants
- $\Gamma := I_U \setminus 0_V = I_U \setminus T_U$: constants of U not in V
- $U: U \rightarrow V$ forgets Γ and the identities $Z_U \setminus Z_V$

Remark

U is faithful, conservative, right adjoint to $F: V \rightarrow U$ (freely adds the constants of Γ):

$$\begin{array}{ccc} & F & \\ & \curvearrowright & \\ U & \xrightarrow[U]{} & V \\ & \perp & \\ & \curvearrowleft & \end{array}$$

This is the adjunction associated with $I_U \rightarrow T_U \iff \eta: 1_V \Rightarrow UF$ is **cartesian**.

▷ It follows that U is a protomodular variety, and therefore an ideally exact category.

Theorem (M. Mancini, F. P.)

The ideally exact context

$$\begin{array}{ccc} & F & \\ & \curvearrowleft & \\ U & \xrightarrow[U]{} & V \\ & \perp & \\ & \curvearrowright & \end{array}$$

is BAT if and only if

$$\omega_A = \omega(\omega_A, \dots, \omega_A, s(c_1^B), \dots, s(c_q^B)),$$

for any coherent split epimorphism $(A, U(B), p, s)$, for any identity $\omega = 0$ in $Z_U \setminus Z_V$ and for every $a_1, \dots, a_p \in A$, where ω_A denotes the element $\omega(a_1, \dots, a_p, s(c_1^B), \dots, s(c_q^B)) \in A$.

Theorem (M. Mancini, G. Metere, F. P.)

The ideally exact context $\mathbf{BWHoops} \xrightarrow[U]{M} \mathbf{WHoops}$ is BAT.

A counterexample

Let **BoolGrp** be the variety of Boolean groups, i.e. abelian groups satisfying

$$x + x = 0.$$

Let

BoolGrp $[u]$

be the variety obtained by adding one unary operation symbol u , with no identities imposed on u . Thus an algebra is a Boolean group A together with an arbitrary function

$$u^A : A \rightarrow A.$$

The operation u^A is not required to be a group homomorphism.

▷ The variety = **BoolGrp** $[u]$ is ideally exact.

Theorem (M. Abbadini, Z. Janelidze, G. Metere, F. P.)

*The canonical ideally exact context associated with **BoolGrp** $[u]$ is not BAT.*

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Thank you for your attention!