# On actions and split extensions in varieties of hoops

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Joint work with Giuseppe Metere and Federica Piazza

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- Let $B$ and $X$ be groups. An *external action* of $B$ on $X$ is a map

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such that

$$(bb') * x = b * (b' * x), \quad 1_B * x = x, \quad b * (xx') = (b * x)(b * x').$$

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- The *semidirect product* $X \rtimes_\xi B$ is $X \times B$ with multiplication

$$(x,b) \cdot (x',b') = (x(b * x'), bb'),$$

where $1_{X \rtimes B} = (1_X, 1_B)$ and $(x,b)^{-1} = (b^{-1} * x^{-1}, b^{-1})$.

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- We can associate with $\xi$ the split extension

$$\begin{tikzcd} X \arrow[r, "\iota_1"] & X \rtimes_\xi B \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "\iota_2"] \end{tikzcd}$$

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$$B \times X \to X\colon (b, x) \mapsto k^{-1}(s(b)k(x)s(b)^{-1}).$$

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- There is a bijection $\mathrm{EAct}(B, X) \cong \mathrm{SplExt}(B, X)$.
- This bijection extends to a natural isomorphism of functors

$$\mathrm{EAct}(-, X) \cong \mathrm{SplExt}(-, X).$$

- This construction can be generalised to any *category of groups with operations*, such as rings, associative algebras, Lie algebras, Leibniz algebras, Jordan algebras, ...

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- Let $B$ and $X$ be associative algebras over a field. An *external action* of $B$ on $X$ consists of a pair of bilinear maps

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$$b * (xx') = (b * x)x', \quad (x * b)x' = x(b * x'), \quad (xx') * b = x(x' * b),$$

$$(bb') * x = b * (b' * x), \quad (b * x) * b' = b * (x * b'), \quad x * (bb') = (x * b) * b'.$$

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- **Can we do the same for other algebraic structures?**

# Hoops

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## Definition

A *hoop* is an algebra $H = (H, \cdot, \to, 1)$ of type $(2, 2, 0)$ such that:

H1. $(H, \cdot, 1)$ is a commutative monoid;

H2. $x \to x = 1$;

H3. $x \cdot (x \to y) = y \cdot (y \to x)$;

H4. $(x \cdot y) \to z = x \to (y \to z)$,

for any $x, y, z \in H$.

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- $1 \to x = x$ and $x \to 1 = 1$.

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- a *product hoop* if it is a basic hoop satisfying

$$(y \to z) \vee ((y \to (x \cdot y)) \to x) = 1,$$

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A *commutative integral residuated lattice* (or CIRL) is an algebra $A = (A, \vee, \wedge, \cdot, \to, 1)$ of type $(2, 2, 2, 2, 0)$ such that:

(1) $(A, \cdot, 1)$ is a commutative monoid.

(2) $(A, \vee, \wedge, 1)$ is a lattice with top element 1.

(3) The following *residuation property* holds in $A$:

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## Definition

A *bounded* CIRL (or BCIRL) is a CIRL with an extra constant 0 that is the bottom element in the lattice order.

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(2) $x \wedge y = x \cdot (x \to y)$, for any $x, y \in A$. (*divisibility condition*)

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- a *Gödel algebra* if it satisfies *idempotency*: $x \cdot x = x$.
- a *product algebra* if it satisfies the identity

$$\neg x \vee ((x \to x \cdot y) \to y) = 1.$$

# Examples

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1. The algebra

$$[0,1]_{\mathrm{MV}} = ([0,1], \cdot_L, \to_L, \max, \min, 0, 1),$$

where $x \cdot_L y = \max\{x + y - 1, 0\}$ and $x \to_L y = \min\{1 - x + y, 1\}$, is called the *standard MV-algebra*.

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2. The algebra

$$[0,1]_G = ([0,1], \cdot_G, \to_G, \max, \min, 0, 1),$$

where $x \cdot_G y = \min\{x, y\}$ and

$$x \to_G y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise} \end{cases}$$

is called the *standard Gödel algebra*.

## Examples

3. The algebra

$$[0,1]_P = ([0,1], \cdot_P, \to_P, \max, \min, 0, 1),$$

where $x \cdot_P y = xy$ and

$$x \to_P y = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{y}{x}, & \text{otherwise} \end{cases}$$

is called the *standard product algebra*.

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- Bounded Wajsberg hoops, bounded Gödel hoops, and bounded product hoops are term-equivalent, respectively, to MV-algebras, Gödel algebras, and product algebras.
- The varieties $\mathsf{BHoops}, \mathsf{WHoops}, \mathsf{GHoops}$ and $\mathsf{PHoops}$ are *semi-abelian* categories.
- The varieties $\mathsf{BLAlg}, \mathsf{MVAlg}, \mathsf{GAlg}$ and $\mathsf{PAlg}$ are *ideally exact* categories, and their initial object is the two-element Boolean algebra $\{0, 1\}$.

# Split extensions with strong section

**Definition (W. Rump, 2009)**

A split extension

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

in the variety $\mathsf{Hoops}$ has *strong section* if $a \to s(b) = sp(a) \to s(b)$.

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## Example (Double negation)

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## Example (Double negation)

Let $A$ be a BL-algebra. The split extension

$$\mathrm{D}(A) \xrightarrow{i} A \underset{j}{\overset{p}{\rightleftarrows}} \mathrm{MV}(A)$$

where $\mathrm{MV}(A) = \{x \in A \mid \neg\neg x = x\}$, $\mathrm{D}(A) = \{x \in A \mid \neg\neg x = 1\}$, $p(a) = \neg\neg a$, and $i$ and $j$ are canonical inclusions, has strong section.

# Strong external actions

**Definition (M. M., G. Metere, F. Piazza, 2026)**

Let $B, X$ be hoops. A *strong external action* of $B$ on $X$ consists of a pair of maps

$$f\colon B \times X \to X\colon (b,x) \mapsto f_b(x),$$
$$g\colon B \times X \to X\colon (b,x) \mapsto g_b(x)$$

such that:

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such that:

E1. $f_b(1) = g_b(1) = 1$;

E2. $f_1 = g_1 = \mathrm{id}_X$;

E3. $f_{b_1 \cdot b_2}(x \cdot g_{b_1}(x \to y)) = f_{b_1 \cdot b_2}(x \cdot (x \to y))$;

E4. $g_{(b_3 \to (b_1 \cdot b_2))}(f_{b_1 \cdot b_2}(x \cdot y) \to f_{b_3}(z))$
$= g_{(b_2 \to b_3) \to b_1}(x \to g_{b_3 \to b_2}(y \to f_{b_3}(z))$,

for any $b, b_1, b_2, b_3 \in B$ and $x, y, z \in X$.

## Proposition (M. M., G. Metere, F. Piazza, 2026)

*Let $B, X$ be hoops and let $f, g\colon B \times X \to X$ be a strong external action. Then the set*

$$A = \{(x,b) \in X \times B \mid f_b(x) = x\}$$

*endowed with the operations*

$$(x,b) \to (y,b') = (g_{b' \to b}(x \to y), b \to b'),$$
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*and unit $1_A = (1,1)$ is a hoop.*

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$$\begin{aligned}(x, b) \to (y, b') &= (g_{b' \to b}(x \to y), b \to b'),\\ (x, b) \cdot (y, b') &= (f_{b \cdot b'}(x \cdot y), b \cdot b')\end{aligned}$$

*and unit $1_A = (1, 1)$ is a hoop. Furthermore, the split extension*

$$\begin{tikzcd} X \arrow[r, "\iota_1"] & A \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "\iota_2"] \end{tikzcd}$$

*where $\iota_1(x) = (x, 1)$, $\iota_2(b) = (1, b)$ and $\pi_2(x, b) = b$, has strong section.*

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*Let*

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B, \arrow[l, shift left, "s"] \end{tikzcd}$$

*be a split extension with strong section in* Hoops. *Then the pair of maps*

$$f, g \colon B \times X \to X$$

*defined by*

$$f_b(x) = s(b) \to (s(b) \cdot x) \quad \textit{and} \quad g_b(x) = s(b) \to x$$

*gives rise to a strong external action of $B$ on $X$.*

**Proposition (M. M., G. Metere, F. Piazza, 2026)**

*Let $B, X$ be hoops. There is a bijection $\tau_B$ between the sets $\mathrm{SplExt}_{\mathrm{ss}}(B, X)$ and $\mathrm{EAct}_{\mathrm{ss}}(B, X)$.*

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## Example (Double negation)

Let $A$ be a BL-algebra. The strong external action associated with the *double negation* split extension is

$$f, g \colon \mathrm{MV}(A) \times \mathrm{D}(A) \to \mathrm{D}(A)$$

defined by

$$f_b(x) = b \to (b \cdot x) \quad \text{and} \quad g_b(x) = b \to x.$$

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- $\mathsf{BHoops} \Rightarrow$

$$g_{\tilde{b}}(g_{b_3 \to (b_1 \to b_2)}(g_{b_2 \to b_1}(x \to y) \to z) \\ \to (g_{b_3 \to (b_2 \to b_1)}(g_{b_1 \to b_2}(y \to x) \to z) \to z)) = 1,$$

where $\tilde{b} = (((b_2 \to b_1) \to b_3) \to b_3) \to ((b_1 \to b_2) \to b_3)$.

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- WHoops $\Rightarrow g_{b_2 \to b_1}(x \to y) \to y = g_{b_1 \to b_2}(y \to x) \to x$.
- PHoops

# What about subvarieties?

- The result extends to the subvarieties of basic, Wajsberg, Gödel and product hoops.
- We have to require more axioms on $(f,g)$.
- BHoops $\Rightarrow$

$$g_{\tilde{b}}(g_{b_3\to(b_1\to b_2)}(g_{b_2\to b_1}(x\to y)\to z)\\ \to (g_{b_3\to(b_2\to b_1)}(g_{b_1\to b_2}(y\to x)\to z)\to z))=1,$$

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- PHoops $\Rightarrow$

$$g_{((b_2\to(b_1b_2))\to b_1)\to(b_2\to b_3)}(g_{b_3\to b_2}(y\to z)\to((y\to f_{b_1b_2}(xy))\to x))\to((y\to f_{b_1b_2}(xy))\to x))\\ \wedge(g_{(b_2\to b_3)\to((b_2\to(b_1b_2))\to b_1)}(((y\to f_{b_1b_2}(xy))\to x)\to g_{b_3\to b_2}(y\to z))\to g_{b_3\to b_2}(y\to z))=1.$$

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- GHoops $\Rightarrow$ **nothing!**

# Thank you!


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