On Bochvar Algebras and Regular Double Stone Algebras

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Bochvar algebras and Płonka sums

Bochvar algebras ([Finn and Grigolia, 1980], [Finn and Grigolia, 1993]) are the algebraic counterpart of external weak Kleene logics. They form a (proper) quasivariety $\mathcal{BCA}$ generated by the algebra $\mathbf{WK}^e = \langle\{1, 1/2, 0\}, \Cap, \Cup, \neg, J_2, 1, 0\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$, whose operations are displayed by the following tables:

$\Cap$	0	$1/2$	1
0	0	$1/2$	0
$1/2$	$1/2$	$1/2$	$1/2$
1	0	$1/2$	1

$\Cup$	0	$1/2$	1
0	0	$1/2$	1
$1/2$	$1/2$	$1/2$	$1/2$
1	1	$1/2$	1

	$\neg$
1	0
$1/2$	$1/2$
0	1

	J_2
1	1
$1/2$	0
0	0

Bochvar algebras have been studied by [Bonzio and Pra Baldi, 2024] through the technique of Płonka sums ([Płonka, 1967], [Płonka and Romanowska, 1992]).

Definition

A *semilattice direct system* of algebras of type τ is a triple $\mathbb{A} = \langle \{\mathbf{A}_i\}_{i \in I}, \langle I, \leq \rangle, \{p_{ij} | i \leq j \in I\} \rangle$ consisting of:

- a semilattice $\langle I, \leq \rangle$ with join $\vee$;
- a family $\{\mathbf{A}_i\}_{i \in I}$ of τ-algebras with pairwise disjoint universes;
- a homomorphism $p_{ij} : \mathbf{A}_i \to \mathbf{A}_j$, for every $i, j \in I$ such that $i \leq j$. Moreover, if $i \leq j, j \leq k$, $p_{ik} = p_{jk} \circ p_{ij}$, and $p_{ii} = id_{A_i}$.

When τ contains constants, $\langle I, \leq \rangle$ becomes a semilattice with zero.

Definition

For a semilattice direct system $\mathbb{A}$, the *Płonka sum* $\mathcal{P}_{ł}(\mathbb{A})$ over $\mathbb{A}$ is the algebra, also of type τ, such that:

- its universe is $\bigcup_{i \in I} A_i$;
- for every n-ary operation $g \in \tau$, and $a_1 \in A_{i_1}, ...a_n \in A_{i_n}$;

$$g^{\mathcal{P}_{ł}(\mathbb{A})}(a_1, ...a_n) := g^{A_j}(p_{i_1 j}(a_1), ...p_{i_n j}(a_n))$$

  where $j = a_{i_1} \vee ...a_{i_n}$;
- with constants in the type: $c^{\mathcal{P}_{ł}(\mathbb{A})} \in A_{i_0}$.

An *involutive bisemilattice* is an algebra $\mathbf{B} = \langle B, \Cap, \Cup, \neg, 1, 0 \rangle$ which can be represented as a Płonka sum whose fibers are Boolean algebras. They form the variety $\mathcal{IBSL}$. By the property of Płonka sums, an equation $\alpha \approx \beta$ is valid in $\mathcal{IBSL}$ iff it is valid in $\mathcal{BA}$ and it is regular, i.e. $Var(\alpha) = Var(\beta)$.

Fact ([Bonzio and Pra Baldi, 2024], Prop 3.5)

Every Bochvar algebra has a $\mathcal{IBSL}$-reduct that satisfies
$x \approx \neg x \ \& \ y \approx \neg y \Rightarrow x \approx y$.

Theorem ([Bonzio and Pra Baldi, 2024])

Let $\mathbf{A}$ *be a Bochvar algebra with* $\mathcal{P}_{\not{l}}(A_i)_{i \in I}$ *the Płonka sum representation of its* $\mathcal{IBSL}$*-reduct. Then:*

(i) *for every* $i \in I$ *and* $a \in A_i$, $J_2(a) \in p_{0i}^{-1}(a)$*; in particular for* $a \in A_0, J_2(a) = a$*;*

(ii) *for every* $i \in I$, $J_2 : \mathbf{A}_i \to [\mathbf{0}, \mathbf{J_2 1_i}]$ *is a (Boolean) isomorphism (with inverse the restriction of* p_{0i} *to* $J_2[\mathbf{A}_i]$*), with* $[\mathbf{0}, \mathbf{J_2 1_i}] := \langle [0, J_2 1_i], \vee, \wedge, ^*, 0, J_2 1_i \rangle$ *an interval Boolean algebra, where negation* * *is defined as* $a^* := \neg a \wedge J_2 1_i$*;*

(iii) *for every* $i \neq j$, $J_2 1_i \neq J_2 1_j$*; in particular, if* $i < j$ *then* $J_2 1_j < J_2 1_i$.

Observe that the J_2 operator breaks the structure of a Płonka sum, since for all $a, J_2(a) \in A_0$, yet Bochvar algebras have a Płonka-style representation.

Bochvar systems and categorical equivalence

The previous theorem showed a core property of the structure theory of $\mathcal{BCA}$: the presence of a Boolean subalgebra which encodes the information of the entire algebra in the form of isomorphic copies of all the fibers.
This has been expressed by [Bonzio et al., 2024] using the notion of Bochvar system:

Definition

A *Bochvar system* is a pair $\mathbb{B} = \langle \mathbf{B}, \mathbf{S} \rangle$ such that $\mathbf{B} = \langle B, \wedge, \vee, \neg, 1, 0 \rangle$ is a Boolean algebra and $\mathbf{S} = \langle S, \wedge, 1 \rangle$ is a meet-subsemilattice with unit of $\mathbf{B}$. In particular, a Bochvar system $\langle \mathbf{B}, \mathbf{I} \rangle$ is called *filtered* if the semilattice $\mathbf{I}$ is a filter of $\mathbf{B}$.

Bochvar algebras and Bochvar systems are essentially the same structures, since they are equivalent categories ([Bonzio et al., 2024, Th.16]).

For $b \in B$, by $[b)$ we denote the principal lattice filter generated by b in $\mathbf{B}$, and for F a filter of $\mathbf{B}$, by $\mathbf{B}/F$ we denote the quotient $\mathbf{B}/\theta_F$, where

$$\theta_F = \{\langle a, b\rangle \in B^2 : (\neg a \vee b) \wedge (\neg b \vee a) \in F\}.$$

If $\mathbb{B} = \langle \mathbf{B}, \mathbf{I}\rangle$ is a Bochvar system, let

$$\mathbb{A} = \langle \{\mathbf{A}_i\}_{i\in I}, \mathbf{I}^{\partial}, \{p_{ij} : i \leq_{\mathbf{I}} j\}\rangle,$$

be such that:

- for all $i \in I$, $\mathbf{A}_i := \mathbf{B}/[i)$;
- $\mathbf{I}^{\partial}$ is the semilattice dual to $\mathbf{I}$;
- for all $i, j \in I$ such that $i \leq_{\mathbf{I}^{\partial}} j$, $p_{ij}(a/[i)) := (a/[i))/[j)$.

It can be proved that $\mathcal{P}_{\nmid}(\mathbb{A})$ is the involutive bisemilattice reduct of a unique Bochvar algebra $\mathbf{A}_{\mathbb{B}}$.

In the other direction, we start with a Bochvar algebra

$$\mathbf{A} = \langle A, \Cap, \Cup, \neg, J_2, 1, 0\rangle$$

whose involutive bisemilattice reduct decomposes as $\mathcal{P}_{\nmid}(\mathbf{A}_i)_{i\in I}$. We define $\mathbb{B}_{\mathbf{A}} := \langle \mathbf{A}_{i_0}, \mathbf{K}\rangle$, where:

- $K = \{J_2^{\mathbf{A}}(1_i) : i \in I\}$;
- for $i, j \in I$, $J_2^{\mathbf{A}}(1_i) \leq_{\mathbf{K}} J_2^{\mathbf{A}}(1_j)$ iff $j \leq_{\mathbf{I}} i$.

We obtain that $\mathbb{B}_{\mathbf{A}}$ is a Bochvar system.

The above maps are mutually inverse and can be lifted to a full-fledged categorical equivalence. Indeed, let:

- $\mathfrak{B}$ denote the category of Bochvar algebras;
- $\mathfrak{S}$ denote the category whose objects are Bochvar systems, and such that, if $\mathbb{B}_1 = \langle \mathbf{B}_1, \mathbf{I}_1\rangle$ and $\mathbb{B}_2 = \langle \mathbf{B}_2, \mathbf{I}_2\rangle$ are objects in $\mathfrak{S}$, a morphism from $\mathbb{B}_1$ to $\mathbb{B}_2$ is a homomorphism g from $\mathbf{B}_1$ to $\mathbf{B}_2$ such that $g(i) \in I_2$ for every $i \in I_1$.

We define the functor $\Gamma : \mathfrak{B} \to \mathfrak{S}$ as follows:

- If $\mathbf{A}$ is an object in $\mathfrak{B}$, $\Gamma(\mathbf{A}) := \mathbb{B}_{\mathbf{A}}$.
- If $f : \mathbf{A}_1 \to \mathbf{A}_2$ is a morphism in $\mathfrak{B}$, $\Gamma(f)$ is the restriction of f to $\mathbf{A}_{1_{i_0}}$.

Similarly, we define the map $\Xi : \mathfrak{S} \to \mathfrak{B}$ as follows:

- If $\mathbb{B}$ is an object in $\mathfrak{S}$, $\Xi(\mathbb{B}) := \mathbf{A}_{\mathbb{B}}$.
- If $g : \mathbb{B}_1 \to \mathbb{B}_2$ is a morphism in $\mathfrak{S}$, $\Xi(g)$ is defined as follows: $\Xi(g)(a/[i)) := g(a)/[g(i))$.

Theorem ([Bonzio et al., 2024], Th.16)

Γ and Ξ are functors that induce a categorical equivalence between the categories $\mathfrak{B}$ and $\mathfrak{S}$.

Regular double Stone algebras

Definition

A *Stone algebra* is an algebra $\mathbf{L}^{\sim} = \langle L^{\sim}, \sqcap, \sqcup, {}^{\sim}, 1, 0\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ which is a bounded distributive lattice with a unary operation ${}^{\sim}$ satisfying:

- $a \sqcap b = 0$ if and only if $a \leq b^{\sim}$;
- $a^{\sim} \sqcup a^{\sim\sim} = 1$.

A *dual Stone algebra* is an algebra is an algebra $\mathbf{L}^{+} = \langle L^{+}, \sqcap, \sqcup, {}^{+}, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ which is a bounded distributive lattice with a unary operation ${}^{+}$ satisfying:

- $a \sqcup b = 1$ if and only if $a \geq b^{+}$;
- $a^{+} \sqcap a^{++} = 0$.

A *regular double Stone algebra* is an algebra $\mathbf{L} = \langle L, \sqcap, \sqcup, {}^{\sim}, {}^{+}, 1, 0\rangle$ s.t. $\langle L, \sqcap, \sqcup, {}^{\sim}, 1, 0\rangle$ is a Stone algebra, $\langle L, \sqcap, \sqcup, {}^{+}, 1, 0\rangle$ is a dual Stone algebra and for all $a, b \in L$, if $a^{\sim} = b^{\sim}$ and $a^{+} = b^{+}$ then $a = b$.

Regular double Stone algebras form the variety $\mathcal{RDSA}$ ([Katriňák, 1973b]). The variety is generated by $\mathbf{SK}^e = \langle \{1, 1/2, 0\}, \sqcap, \sqcup, ^{\sim}, ^{+}, 1, 0 \rangle$:

	$\sim$
1	0
1/2	0
0	1

	$+$
1	0
1/2	1
0	1

$\sqcap$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$\sqcup$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

Given a regular double Stone algebra $\mathbf{L}$, we denote the set of its *sharp* elements by:

$$S(\mathbf{L}) := \{x \in L : x = x^{\sim\sim}\}.$$

Notice that $S(\mathbf{L})$ is the largest Boolean subuniverse of $\mathbf{L}$. Moreover, in $S(\mathbf{L})$ the operations $^{+}$ and $^{\sim}$ coincide, and therefore are antitone involutions.

We denote the set of *dense* and *dually dense* elements of $\mathbf{L}$ respectively by:

$$D(\mathbf{L}) = \{x \in L : x^{\sim} = 0\},$$

$$DD(\mathbf{L}) = \{x \in L : x^{+} = 1\}.$$

Every regular double Stone algebra admits a representation in terms of a Boolean algebra endowed with additional operations (see [Düntsch, 1997, Katriňák, 1973a]). This result can be restated in an equivalent variant using a twist product construction (originally developed for Nelson algebras in [Fidel, 1978, Vakarelov, 1977]).

Definition

Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{F} \rangle$ be a filtered Bochvar system. The *twist product algebra* over $\mathbb{B}$ is the algebra $Tw(\mathbb{B}) = \langle T, \sqcap, \sqcup, ^{\sim}, ^{+}, 1, 0 \rangle$ of type $\langle 2, 2, 1, 1, 0, 0 \rangle$, such that:

- $T := \{\langle a, b \rangle : a, b \in B, a \wedge b = 0, a \vee b \in F\}$;
- $\langle a, b \rangle \sqcap \langle c, d \rangle := \langle a \wedge c, b \vee d \rangle$;
- $\langle a, b \rangle \sqcup \langle c, d \rangle := \langle a \vee c, b \wedge d \rangle$;
- $\langle a, b \rangle^{\sim} := \langle b, \neg b \rangle$;
- $\langle a, b \rangle^{+} := \langle \neg a, a \rangle$;
- $0 := \langle 0, 1 \rangle$;
- $1 := \langle 1, 0 \rangle$.

Theorem

Every regular double Stone algebra is of the form $Tw(\mathbb{B})$, for a unique filtered Bochvar system $\mathbb{B} = \langle S(\mathbf{A}), h(D(\mathbf{A}))\rangle$, where $h : D(\mathbf{A}) \to S(\mathbf{A})$ is defined by $h(x) = x^{++}$.

Let $\mathfrak{RDSA}$ denote the category of regular double Stone algebras, while $\mathfrak{F}$ denote the category whose objects are filtered Bochvar systems and a morphism from $\mathbb{B}_1$ to $\mathbb{B}_2$ is a homomorphism g s.t. $g(i) \in F_2$ for every $i \in F_1$.

We define the functor $Tw : \mathfrak{F} \to \mathfrak{RDSA}$ as follows:

- if $\mathbb{B}$ is an object in $\mathfrak{F}$, $Tw(\mathbb{B}) := Tw(\mathbb{B})$;
- if $f : \mathbb{B}_1 \to \mathbb{B}_2$ is a morphism in $\mathfrak{F}$, we define $Tw(f)(\langle a, b\rangle) := \langle f(a), f(b)\rangle$.

Similarly, we define the functor $\Lambda : \mathfrak{RDSA} \to \mathfrak{F}$ as follows:

- if $\mathbf{A}$ is an object in $\mathfrak{RDSA}$, $\Lambda(\mathbf{A}) := \langle S(\mathbf{A}), h(D(\mathbf{A}))\rangle$;
- if $g : \mathbf{A}_1 \to \mathbf{A}_2$ is a morphism in $\mathfrak{RDSA}$, $\Lambda(g)$ is the restrictions of g to $S(\mathbf{A_1})$.

Λ and Tw are induce a categorical equivalence between $\mathfrak{RDSA}$ and $\mathfrak{F}$ ([Düntsch, 1997, Katriňák, 1973a]).

Modified twist product

The functor $\Xi : \mathfrak{S} \to \mathfrak{B}$ can be realised via a modified twist product construction, avoiding Płonka sums entirely.

Definition

Let $\mathbb{B} = \langle \mathbf{B}, \mathbf{I} \rangle$ be a Bochvar system. The *modified twist product algebra* over $\mathbb{B}$ is the algebra $Tw^{+}(\mathbb{B}) = \langle T, \Cap, \Cup, \neg, J_2, 1, 0 \rangle$ of type $\langle 2, 2, 1, 1, 0, 0 \rangle$, such that:

- $T := \{\langle a, b \rangle : a, b \in B, a \wedge b = 0, a \vee b \in I\}$;
- $\langle a, b \rangle \Cap \langle c, d \rangle := \langle a \wedge c, (b \wedge d) \vee (b \wedge c) \vee (a \wedge d) \rangle$;
- $\langle a, b \rangle \Cup \langle c, d \rangle := \langle (a \wedge c) \vee (a \wedge d) \vee (b \wedge c), b \wedge d \rangle$;
- $\neg \langle a, b \rangle := \langle b, a \rangle$;
- $J_2 \langle a, b \rangle := \langle a, \neg a \rangle$;
- $0 := \langle 0, 1 \rangle$;
- $1 := \langle 1, 0 \rangle$.

Theorem

$Tw^+(\mathbb{B})$ is a Bochvar algebra isomorphic to $\mathbf{A}_{\mathbb{B}}$. The functors Γ and Tw^+ induce a categorical equivalence between $\mathfrak{B}$ and $\mathfrak{S}$.

For filtered Bochvar systems, the modified twist product decomposes into the standard twist product and a translation from the language of $\mathcal{RDSA}$ to $\mathcal{BCA}$.

Definition

For $\mathbf{S} \in \mathcal{RDSA}$, define $\mathcal{B}(\mathbf{S}) := \langle S, \Cap, \Cup, \neg, J_2, 1, 0\rangle$ by:

$$\neg a := a^{\sim} \sqcup (a \sqcap a^{+}), \quad a \Cap b := (a \sqcap b) \sqcup (a \sqcap \neg a) \sqcup (b \sqcap \neg b),$$

$$a \Cup b := (a \sqcup b) \sqcap (a \sqcup \neg a) \sqcap (b \sqcup \neg b), \quad J_2 a := a^{++}.$$

Theorem

1. *If $\mathbf{S} \in \mathcal{RDSA}$, then $\mathcal{B}(\mathbf{S}) \in \mathcal{BCA}$.*
2. *If $\mathbb{B}$ is a filtered Bochvar system, then $Tw^+(\mathbb{B}) = \mathcal{B}(Tw(\mathbb{B}))$.*

Categorical relations

Given a Bochvar system $\mathbb{B} = \langle \mathbf{B}, \mathbf{I} \rangle$, define its *filter completion* $\uparrow\mathbb{B} = \langle \mathbf{B}, \mathrm{Fg}(I) \rangle$, where $\mathrm{Fg}(I)$ is the filter generated by $\mathbf{I}$ in $\mathbf{B}$.

Theorem

$\mathfrak{F}$ is a full and reflective subcategory of $\mathfrak{S}$.

Reflectivity is proven by showing that the inclusion functor $I : \mathfrak{S} \to \mathfrak{B}$ has a left adjoint in the for of the functor $\Pi : \mathcal{S} \to \mathcal{F}$, defined as $\Pi(\mathbb{B}) = \uparrow\mathbb{B}$ and $\Pi(f) = f$.

Corollary

$\mathfrak{RDSA}$ is equivalent to a full and reflective subcategory of $\mathfrak{B}$. The embedding functor is $\Xi \circ i \circ \Lambda$, with left adjoint $Tw \circ \Pi \circ \Gamma$.

The inclusion I is not essentially surjective though, so $\mathcal{RDSA}$ is a proper subcategory of $\mathcal{B}$.

Example: The 5-element Bochvar algebra $\mathbf{A}$ whose Płonka representation has:

- bottom fiber: the 4-element Boolean algebra $\mathbf{B}_2 \times \mathbf{B}_2$;
- top fiber: the trivial Boolean algebra.

corresponds to the Bochvar system $\langle \mathbf{B}_2 \times \mathbf{B}_2, \{\langle 1,1\rangle, \langle 0,0\rangle\}\rangle$.

The set $\{\langle 1,1\rangle, \langle 0,0\rangle\}$ is a meet-subsemilattice with 1, but not a filter of $\mathbf{B}_2 \times \mathbf{B}_2$.

Hence $\mathbf{A}$ has no preimage in $\mathfrak{F}$ and is not isomorphic to any $\mathcal{RDSA}$.

Finally we show in which sense every Bochvar algebra can be seen as a colimit of regular double Stone algebras. First we prove that all 1-generated subalgebras of a Bochvar algebra arise from regular double Stone algebras.

Lemma

Let $\mathbf{A} \in \mathcal{BCA}$ *and* $a \in A$. *Then* $\mathrm{Sg}^{\mathbf{A}}(a)$ *is isomorphic to* $\mathcal{B}(\mathbf{S})$ *for some* $\mathbf{S} \in \mathcal{RDSA}$.

Corollary

Every Bochvar algebra is a union of subalgebras that arise from $\mathcal{B}(\mathcal{RDSA})$.

Let $\mathcal{B}^+$ be the category with Bochvar algebras as objects and morphisms preserving constants, unary operations, and meets/joins of the form $a \Cap \neg a$, $a \Cup \neg a$.

Theorem

Every object of $\mathcal{B}^+$ *is the algebra-reduct of a colimit of a diagram in* $\mathcal{RDSA}$.

$$
\begin{tikzcd}
\mathfrak{F} \arrow[rr, "Tw"] \arrow[d, "I"'] & & \mathfrak{RDSA} \arrow[ll, bend left, "\Lambda"] \arrow[d, "\mathcal{B}"] \\
\mathfrak{S} \arrow[rr, "\Xi = Tw^{+}"] & & \mathfrak{B} \arrow[ll, bend left, "\Gamma"]
\end{tikzcd}
$$

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