A Goldblatt-Thomason Theorem for Compact Polyhedra

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Hello!

A small presentation,
from least to most important

- My name is **Francesco Tognetti**, 
- I'm a second year PhD student at the **University of Luxembourg**,
- I have a very cute **dog**.

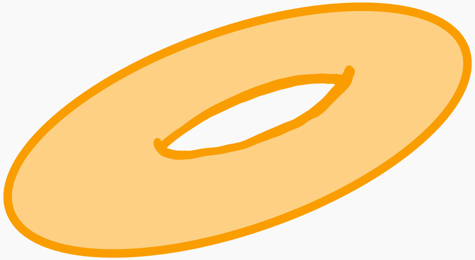
Oopsie

The **abstract** in the conference website said I'd be presenting a Goldblatt-Thomason theorem for **Pre-compact** polyhedra.

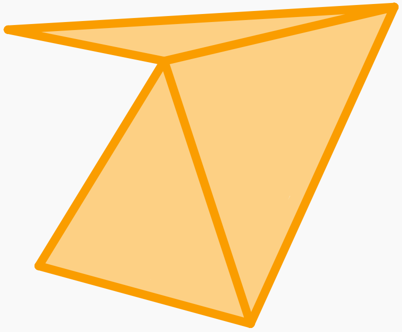
Apparently that approach was **flawed**, to the point that the result turned out not to be valid.

We managed to prove one for **Compact** polyhedra instead.

(I'm very sorry for the mix-up)



Topological and Polyhedral semantics



Topological semantics

A quick recap is in order:

We are doing **Polyhedral semantics of modal logic**, that can be seen as a slight modification of **Topological semantics**.

We have the standard unimodal language

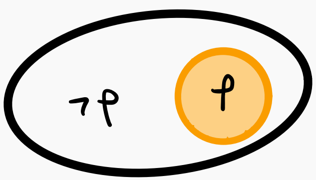
$$\boxed{\mathcal{L} = \{p \in P \mid\mid \bot \mid\mid \neg\varphi \mid\mid \varphi \vee \psi \mid\mid \Diamond \varphi\}}$$

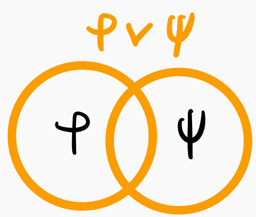
Polyhedral Semantics

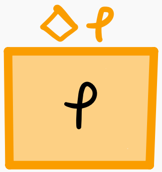
As it's the case in topological semantics we interpret formulas in a space X

$$\nu : \mathcal{L} \to \mathcal{P}(X)$$

- $\perp$ as the **Empty Set** $\quad \nu(\perp) = \emptyset$
- $\neg$ as the **Complementary** $\quad \nu(\neg\varphi) = X \setminus \nu(\varphi)$
- $\vee$ as the **Union** $\quad \nu(\varphi \vee \psi) = \nu(\varphi) \cup \nu(\psi)$
- $\Diamond$ as the **Topological closure** $\quad \nu(\Diamond \varphi) = \overline{\nu(\varphi)}$



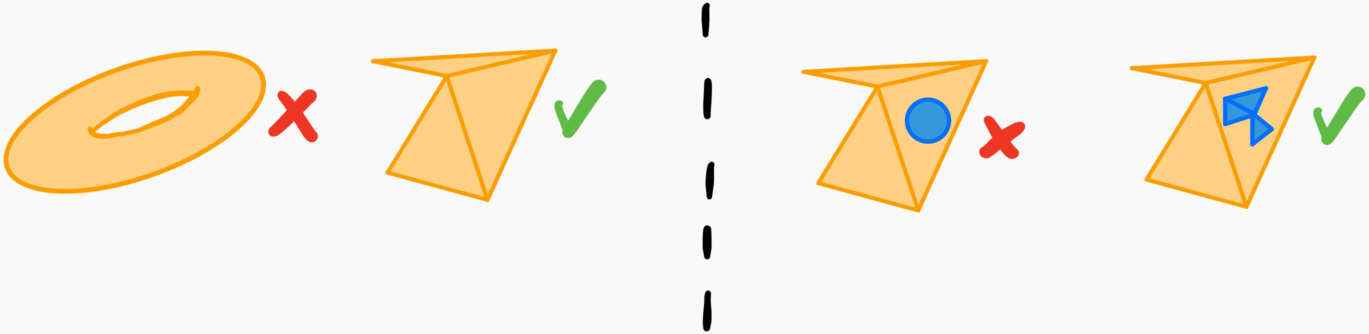




Polyhedral Semantics

The distinguishing factor between **Topological** and **Polyhedral** semantics is that

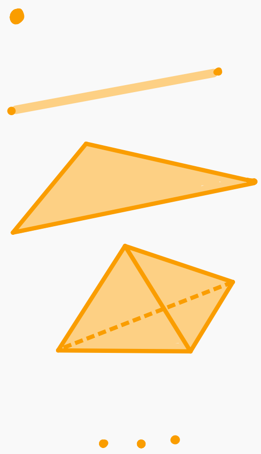
- We require X to be a **Polyhedron**
- For any φ we require $\nu(\varphi)$ to be a **Subpolyhedron**



Polyhedra

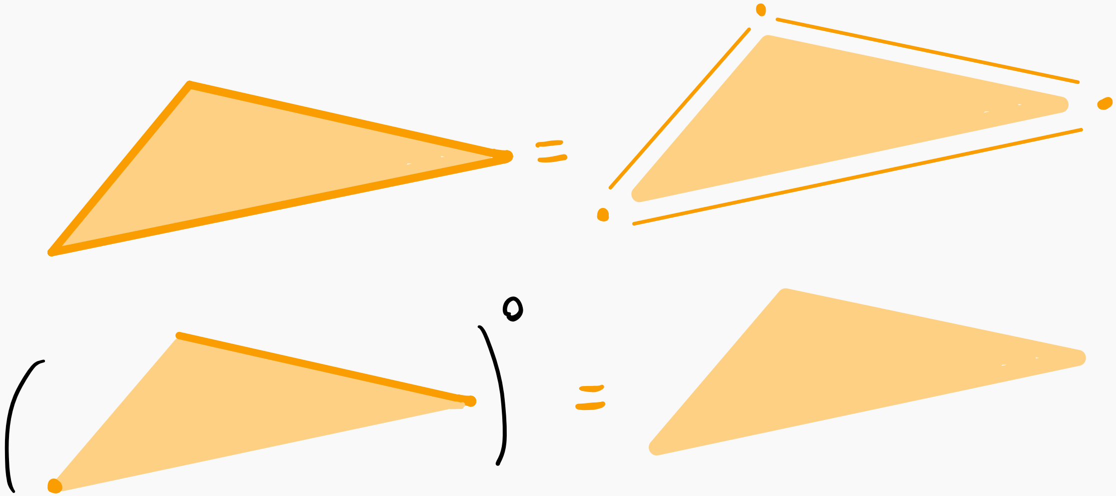
Polyhedra are essentially higher dimensional generalizations of polygons, thus the nicest way to describe them is as **unions of simplices**.

- A $0-$Simplex is a point,
- A $1-$Simplex is a segment,
- A $2-$Simplex is a triangle,
- A $3-$Simplex is a tetrahedron,
- $\ldots$



Polyhedra

Since we want to allow **open** sets (in the euclidean topology)
What we consider as “acceptable subsets” is **unions of relative interior of simplices**



Compactness

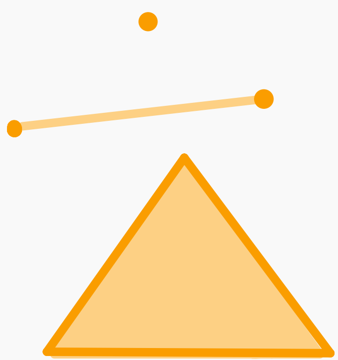
It's not obvious what an **infinite**-polyhedron is: If we allowed them to be arbitrary unions of (relative interiors of) simplices then any topological space is a polyhedron, as every point is a simplex.

There are many possible generalisations but

for now we may restrict to the **compact** case.

This gives rise to a much **tamer** topology:

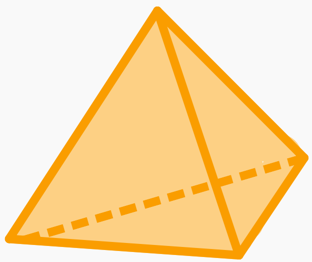
For example every polyhedron of dimension $\leq n$ validates Bd_n (the bounded depth formula of order n)

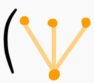
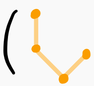


$\models Bd_2$

$\not\models Bd_3$

Or any convex polyhedron validates $\chi($  $) + \chi($  $)$



$\models \mathrm{Bd}_3 + \chi($  $) + \chi($  $)$

$M \models \chi(F) \iff M \not\hookrightarrow F$

A Definability theorem

Definability

From here the **definability** question is pretty natural: Which **conditions** are necessary and sufficient for a class of polyhedra to be **modally definable**?

Theorem (GTKP)
Let $\mathcal{C}$ be a class of compact polyhedra. Then $\mathcal{C}$ is modally definable if and only if it's closed under

(1) ***Finite disjoint unions,*** $\sqcup$

(1) ***Face up-reductions***

Finite disjoint unions

Definition
Let X, Y be two sets, we define the **Disjoint union** of X and Y as the colimit diagram

$$\begin{tikzcd}
 & U & \\
X \arrow[ur] \arrow[r] & X + Y \arrow[u, dashed, "\exists!"] & Y \arrow[ul] \arrow[l]
\end{tikzcd}$$

In the category of sets

Blah blah words to fill out the page because otherwise the visual impact doesn't reflect the joke appropriately du dudududu du du I hear the drums echoing tonight

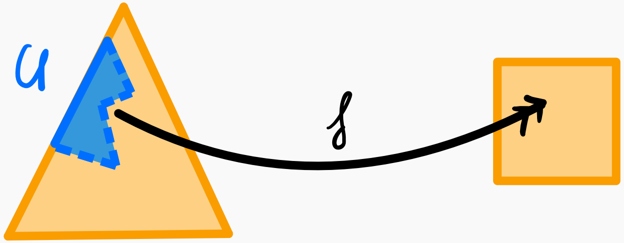
ahah

Just kidding, I know you know

To understand what a face up-reduction is, we start from the concept of topological up-reduction:

Definition (Up-reduction)
Let X, Y be topological spaces, an **up-reduction** $f : X \circ\!\!\longrightarrow\!\!\!\!\rightarrow Y$ Is a **Surjective partial map** such that

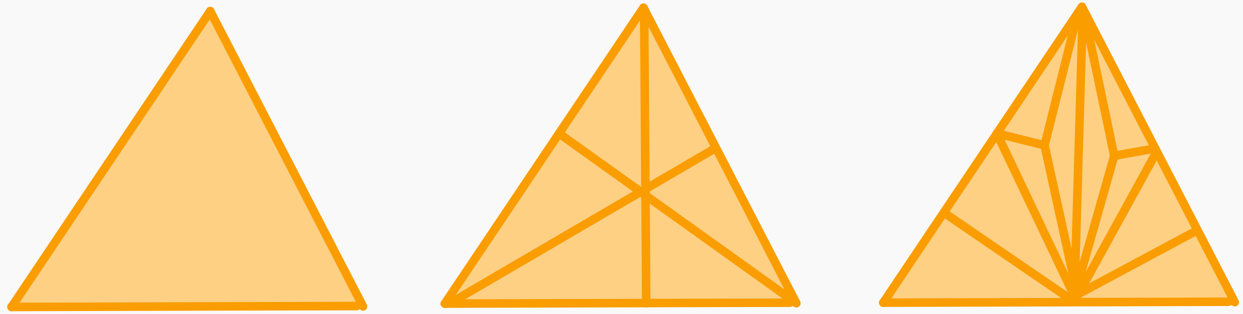
- The **domain** of f is an open subset $U \subseteq^\circ X$,
- $f_{|U}$ is an **open continuous map**.



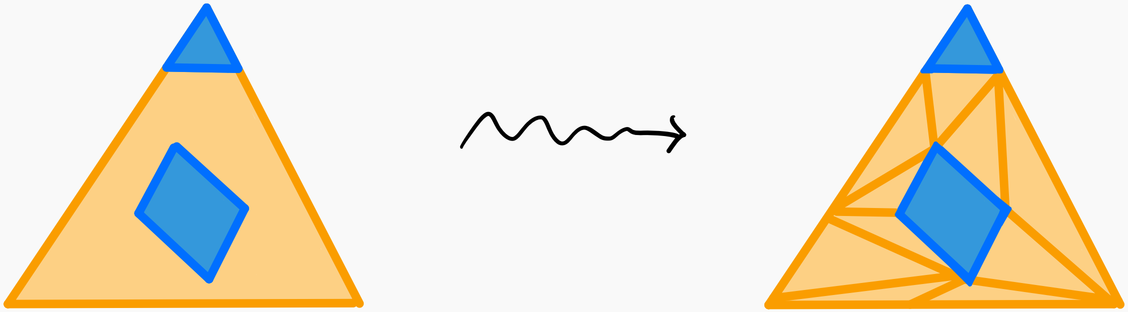
Face posets

The reason why polyhedra are so **well-behaved** is that we can **triangulate** them properly.

That is, for any polyhedron $P \subseteq \mathbb{R}^n$ there are (uncountably) **many** ways of choosing a **triangulation**, i.e. a collection of simplices that cover the polyhedron.



And furthermore, chosen a finite set of subpolyhedra $P_1, \ldots, P_k$ there always exists a finite triangulation of P that triangulates all of them



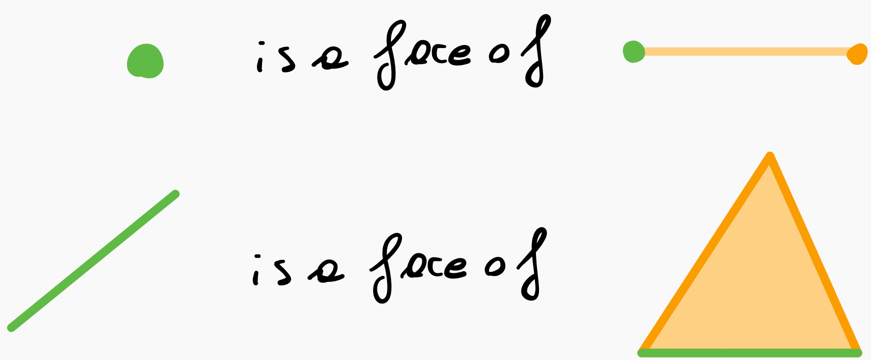
Finiteness

This means that despite the algebra of subpolyhedra $\mathsf{SubPol}(P)$ being infinite,

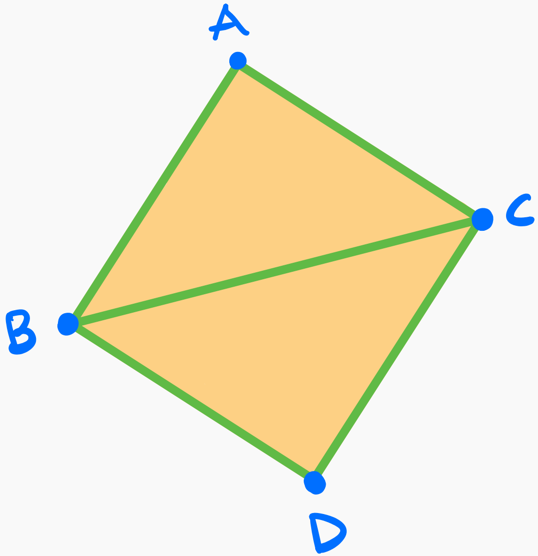
given $\varphi_1, \ldots, \varphi_n$, we can always find a **finite** subalgebra that contains $\{\nu(\varphi_1), \ldots, \nu(\varphi_n)\}$

Posets

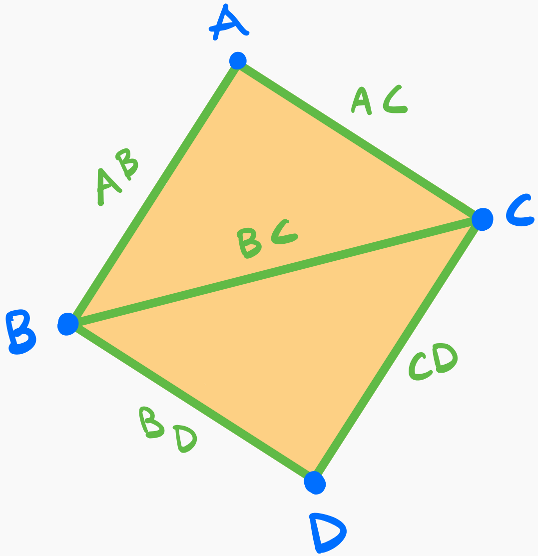
Triangulations have a natural poset structure, given by the **“face of”** relation:

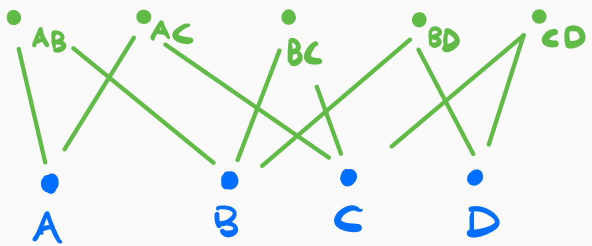


For example

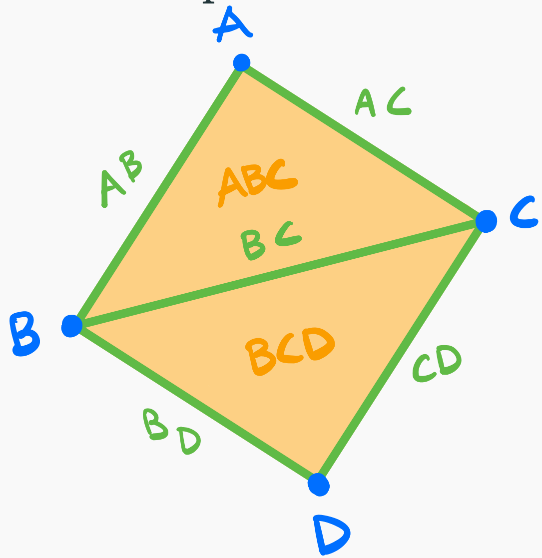


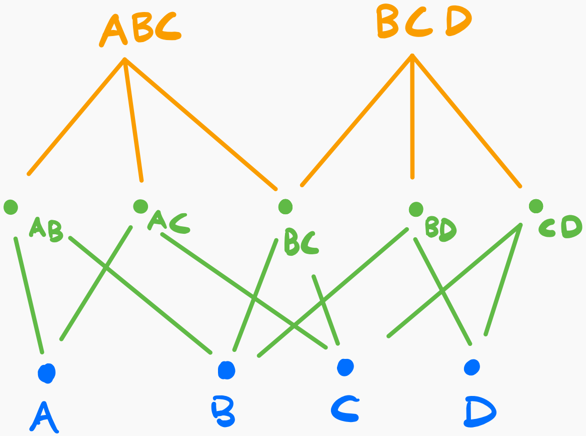
For example





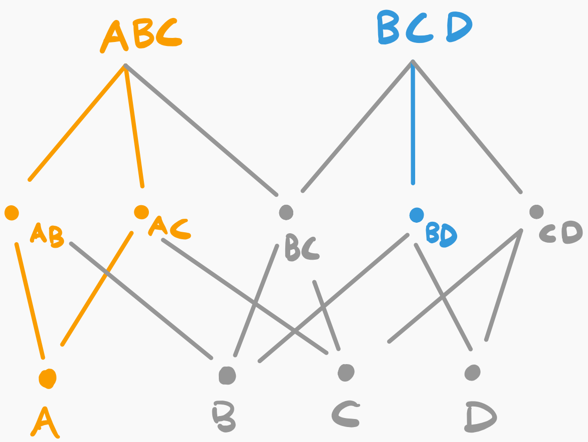
For example



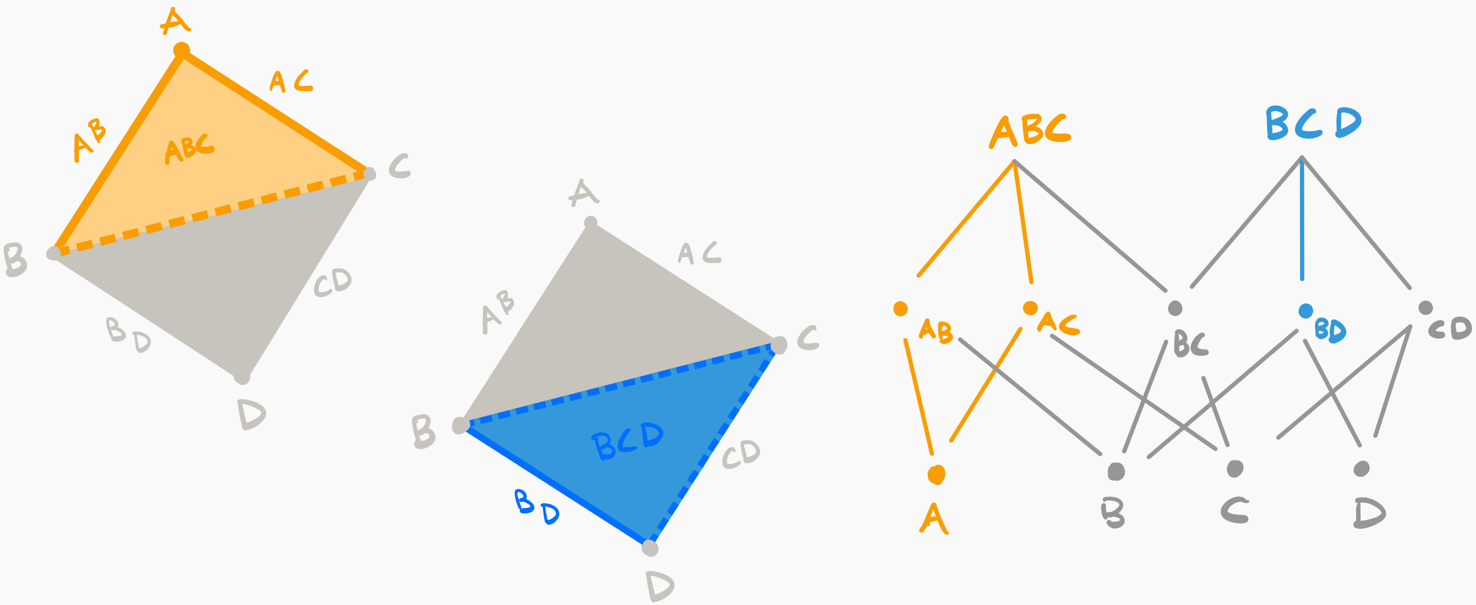


Poset topology

Every poset comes equipped with a topology, where the **upward-closed** sets are open, and the **downward-closed** sets are closed.



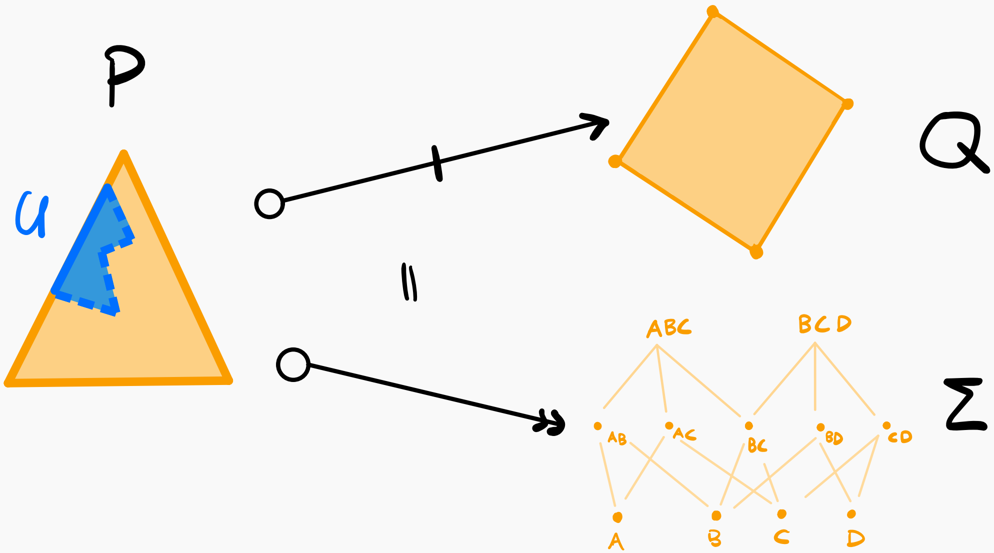
This is compatible with our topology on polyhedra: Given a triangulation T of a polyhedron P, (euclidean) **open subpolyhedra** form upward-closed sets in T



Face up-reductions

Given two polyhedra P and Q a **face up-reduction** $P \circ\!\!\!\longrightarrow\!\!\!\!\!\!\!\!\!+\;\; Q$

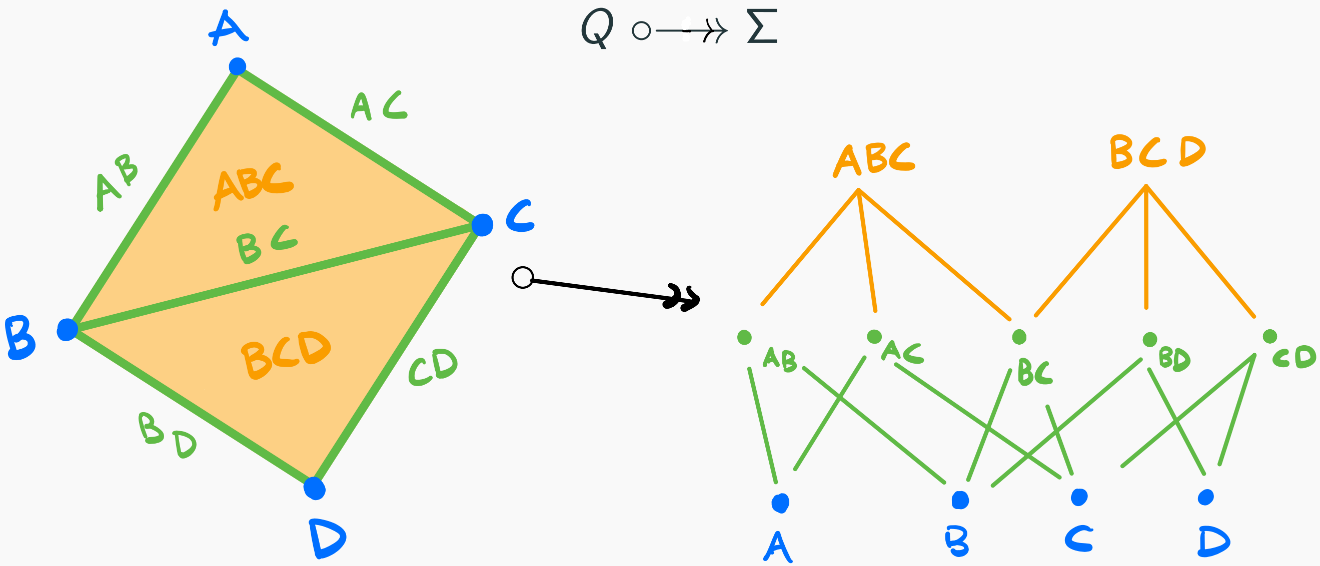
Is a surjective partial map $P \twoheadrightarrow \Sigma$ to a triangulation of Q such that the **preimage** of every open of Σ is an open subpolyhedron of P.



In particular

Given any polyhedron Q and any triangulation Σ, then

$$Q \circ\!\!\longrightarrow\!\!\!\!\!\twoheadrightarrow \Sigma$$



Proof

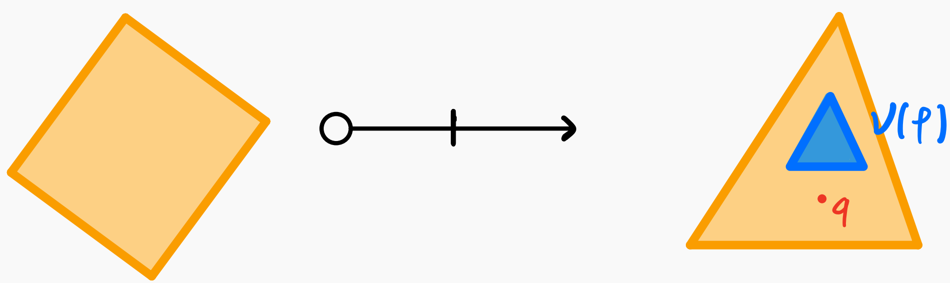
The proof, essentially

Modally definable $\implies$ closed under (1) and (2): Disjoint unions are trivial, let's focus on face up-reductions;

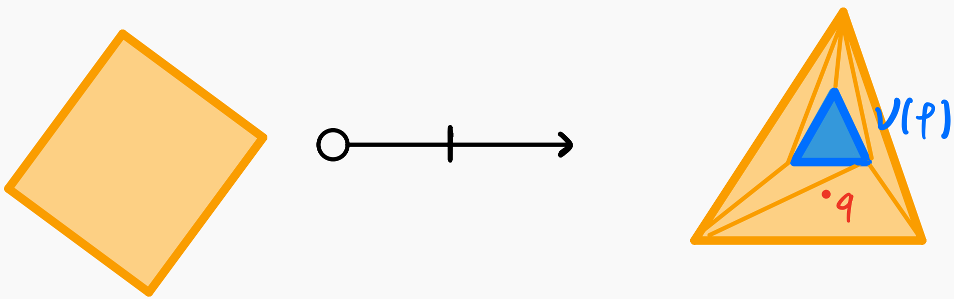
Suppose $P \models F$ and $P \circ\!\!\longmapsto Q$ through a triangulation Σ, then assume $Q \not\models F$

Then there exists a formula $\varphi \in F$ such that $Q \not\models \varphi$

But then, we can find a point q of Q such that $Q, q \not\models \varphi$



This means that there exists a triangulation Σ' that triangulates $\nu(\varphi)$ and subdivides Σ: We can extend $P \circ\!\!\longrightarrow\!\!\!\!\rightarrow \Sigma$ to $P \circ\!\!\longrightarrow\!\!\!\!\rightarrow \Sigma'$



Meaning P refutes φ as well $\rightarrow\!\leftarrow$

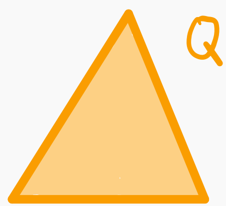
Closed under (1) and (2) $\implies$ modally definable:

Let $\mathcal{C}$ be a class closed under **(1)** and **(2)**.

Call $\mathcal{L}_\mathcal{C}$ the logic of $\mathcal{C}$ and call $\mathcal{C}'$ the polyhedra validating $\mathcal{L}_\mathcal{C}$.

$$\mathcal{L}_\mathcal{C} = \{\varphi : \mathcal{C} \models \varphi\} \rightsquigarrow \mathcal{C}' \in \{P : P \models \mathcal{L}_\mathcal{C}\}$$

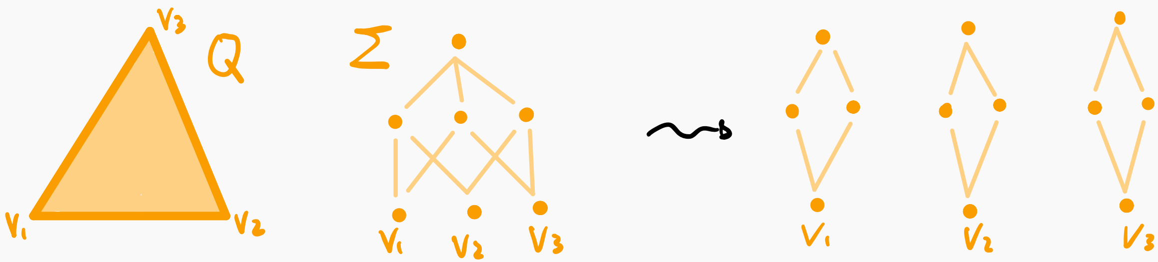
We only need to show that $\mathcal{C}' \subseteq \mathcal{C}$, so let $Q \in \mathcal{C}'$.



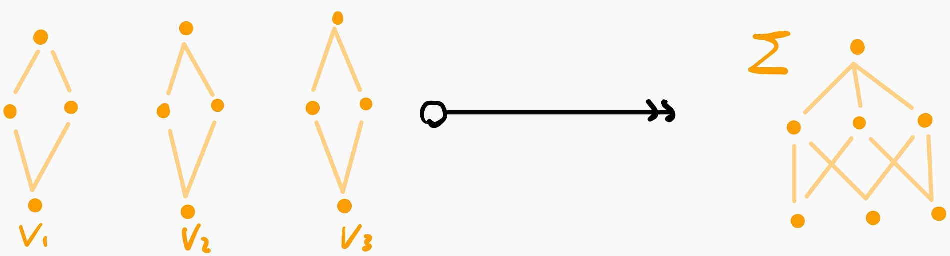
$\models \mathcal{L}$

$Q \models \mathcal{L}$. We want to construct $P \in \mathcal{C}$ such that $P \circ\!\!\twoheadrightarrow Q$.

To do that, we can pick a triangulation Σ of Q, and consider all of its rooted open subposets $(\uparrow v_i)$.

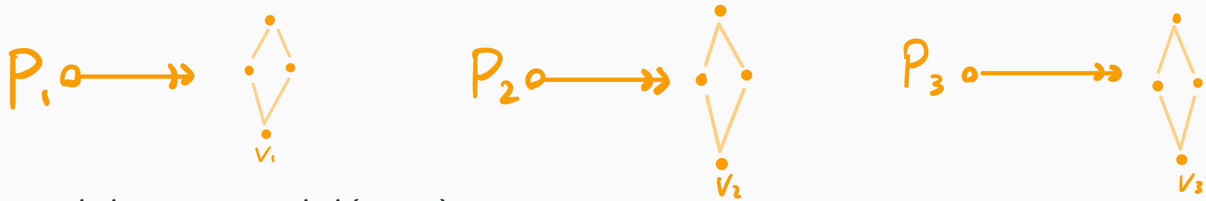


We can pick the Jankov-Fine formulas $\chi_i = \chi(\uparrow v_i)$for each of them, **excluding the presence of an up-reduction to** $(\uparrow v_i)$.



As $(\uparrow v_i)$ are generated subframes of Σ, $\Sigma \not\models \chi_i$ Thus none of the χ_i belong to $\mathcal{L}$

We can find polyhedra $P_i \in \mathcal{C}$ up-reducing to each $(\uparrow v_i)$



Thus $\bigsqcup P_i \circ\!\!\longrightarrow\!\!\!\!\rightarrow \bigsqcup(\uparrow v_i) \circ\!\!\longrightarrow\!\!\!\!\rightarrow \Sigma$

Thus $P := \bigsqcup P_i \circ\!\!\longrightarrow\!\!\!\!\!\!\!\!+\;\; Q$

$\Uparrow$ $\qquad$ $\Uparrow$

$\mathcal{C}$ $\qquad$ $\mathcal{C}$

Boring part over

What can we do with this?

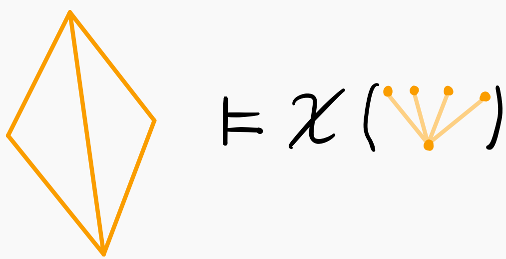
As expected, we can show that the class of polyhedra of bounded dimension is modally definable:

$$\dim(P) \leq n \iff P \models Bd_n$$

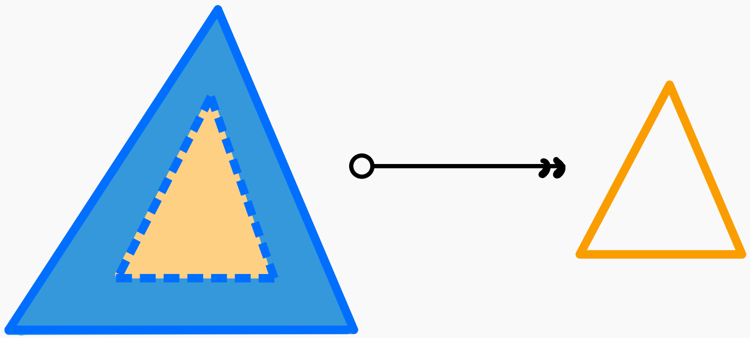
Less expectedly, the class of convex polyhedra is **not** modally definable: There must exist compact non-convex polyhedra that validate the same formulas as all convex ones.

Even more unexpectedly, we can talk about some form of homology in one dimension:

The class of 1d-polyhedra with less than n holes for connected component is modally definable



Somewhat more expectedly, the class of (locally) contractible polyhedra is not modally definable.



future work

There is **some hope** that we may get a result closer to the topological one:

Every **up-reduction** implies **face up-reductions**

$$P \circ\!\!\!\twoheadrightarrow Q \circ\!\!\!\twoheadrightarrow \Sigma \rightsquigarrow P \circ\!\!\!\relbar\!\!\!\!+\!\!\!\!\rightarrow Q$$

And the existence of a **face up-reduction** is independent on the choice of face poset for the same polyhedron.

$$P \circ\!\!\!\relbar\!\!\!\!+\!\!\!\!\rightarrow Q \overset{?}{\rightsquigarrow} P \circ\!\!\!\twoheadrightarrow Q$$

The converse implication is still unknown

It would be interesting to see if applying conditions that generalize the **“less than n holes”** in higher dimensions are possible.

$\mathrm{rk}(H_n(P, G)) \leq n$?

Thank you for listening!

bonus dog pic