Theorems of alternatives, algebraically

Isabel Hortelano Martín

University of Bern

Joint work with George Metcalfe

- Is the following equation valid in $\langle \mathbb{Z}, \wedge, \vee, +, -, 0 \rangle$?

$$0 \leq (x + 4y + 3z) \vee (x - y) \vee (3x + 2y + 3z)$$

- Is the zero vector a non-trivial linear combination over $\mathbb{N}$ of

$$\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}?$$

- Is the following equation valid in $\langle \mathbb{Z}, \wedge, \vee, +, -, 0 \rangle$?

$$0 \leq (x + 4y + 3z) \vee (x - y) \vee (3x + 2y + 3z)$$

- Is the zero vector a non-trivial linear combination over $\mathbb{N}$ of

$$\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}?$$

Theorem (cf. Gordan 1873)

Let $t_1, \dots, t_n \in \mathbb{Z}^m$. Then exactly one of the following holds:

(1) *The zero vector is a non-trivial linear combination of $t_1, \dots, t_n$ over $\mathbb{N}$;*

(2) *There exists $y \in \mathbb{Z}^m$ such that $y^T t_i > 0$ for every $i \in \{1, \dots, n\}$.*

Theorem (cf. Gordan 1873)

Let $t_1, \ldots, t_n \in \mathbb{Z}^m$. Then the following are equivalent:

(1) *The zero vector is a non-trivial linear combination of $t_1, \ldots, t_n$ over $\mathbb{N}$;*

(2) *For every $y \in \mathbb{Z}^m$, there exists some $i \in \{1, \ldots, n\}$ such that $y^T t_i \leq 0$.*

Theorem (cf. Gordan 1873)

Let $t_1, \ldots, t_n \in \mathbb{Z}^m$. Then the following are equivalent:

(1) *The zero vector is a non-trivial linear combination of $t_1, \ldots, t_n$ over $\mathbb{N}$.*

(2) $\langle \mathbb{Z}, \wedge, \vee, +, -, 0 \rangle \models 0 \leq t_1 \vee \cdots \vee t_n$.

Theorem (cf. Gordan 1873)

Let $t_1, \ldots, t_n \in \mathbb{Z}^m$*. Then the following are equivalent:*

(1) *The zero vector is a non-trivial linear combination of* $t_1, \ldots, t_n$ *over* $\mathbb{N}$*.*

(2) $\mathsf{AbLG} \models 0 \leq t_1 \vee \cdots \vee t_n$.

- Consider the non-zero integers

$$Z^* := \{a \in \mathbb{Z} \mid a \neq 0\}.$$

- Equip it with the lattice operations corresponding to the standard ordering $\langle \mathbb{Z}, \leq \rangle$ and let $\neg$ be the additive inversion.

- Define the monoidal operation

$$b \cdot a = a \cdot b = \begin{cases} a \wedge b & \text{if } |a| = |b|, \\ a & \text{if } |a| > |b|. \end{cases}$$

- $\mathbb{V}(\langle Z^*, \wedge, \vee, \cdot, \neg, 1 \rangle)$ is the variety SM of **Sugihara monoids**.

- Consider the non-zero integers

$$Z^* := \{a \in \mathbb{Z} \mid a \neq 0\}.$$

- Equip it with the lattice operations corresponding to the standard ordering $\langle \mathbb{Z}, \leq \rangle$ and let $\neg$ be the additive inversion.

- Define the monoidal operation

$$b \cdot a = a \cdot b = \begin{cases} a \wedge b & \text{if } |a| = |b|, \\ a & \text{if } |a| > |b|. \end{cases}$$

- $\mathbb{V}(\langle Z^*, \wedge, \vee, \cdot, \neg, 1 \rangle)$ is the variety SM of **Sugihara monoids**.

- Consider the non-zero integers

$$Z^* := \{a \in \mathbb{Z} \mid a \neq 0\}.$$

- Equip it with the lattice operations corresponding to the standard ordering $\langle \mathbb{Z}, \leq \rangle$ and let $\neg$ be the additive inversion.

- Define the monoidal operation

$$b \cdot a = a \cdot b = \begin{cases} a \wedge b & \text{if } |a| = |b|, \\ a & \text{if } |a| > |b|. \end{cases}$$

- $\mathbb{V}(\langle Z^*, \wedge, \vee, \cdot, \neg, 1 \rangle)$ is the variety SM of **Sugihara monoids**.

- Consider the non-zero integers

$$Z^* := \{a \in \mathbb{Z} \mid a \neq 0\}.$$

- Equip it with the lattice operations corresponding to the standard ordering $\langle \mathbb{Z}, \leq \rangle$ and let $\neg$ be the additive inversion.

- Define the monoidal operation

$$b \cdot a = a \cdot b = \begin{cases} a \wedge b & \text{if } |a| = |b|, \\ a & \text{if } |a| > |b|. \end{cases}$$

- $\mathbb{V}(\langle Z^*, \wedge, \vee, \cdot, \neg, 1 \rangle)$ is the variety SM of **Sugihara monoids**.

- Define

$$a + b = -(-a \cdot -b) = \begin{cases} a \vee b & \text{if } |a| = |b|, \\ a & \text{if } |a| > |b|. \end{cases}$$

Theorem (cf. Avron 1987)

The following are equivalent for any $\{\cdot, \neg, 1\}$- terms $t_1, \ldots, t_n$:

(1) SM $\models 1 \leq \lambda_1 t_1 + \cdots + \lambda_n t_n$ *for some* $\lambda_1, \ldots, \lambda_n \in \{0, 1\}$ *not all* 0.

(2) SM $\models 1 \leq t_1 \vee \cdots \vee t_n$.

- Define

$$a + b = -(-a \cdot -b) = \begin{cases} a \vee b & \text{if } |a| = |b|, \\ a & \text{if } |a| > |b|. \end{cases}$$

Theorem (cf. Avron 1987)

The following are equivalent for any $\{\cdot, \neg, 1\}$*- terms* $t_1, \ldots, t_n$*:*

(1) SM $\models 1 \leq \lambda_1 t_1 + \cdots + \lambda_n t_n$ *for some* $\lambda_1, \ldots, \lambda_n \in \{0, 1\}$ *not all* 0.

(2) SM $\models 1 \leq t_1 \vee \cdots \vee t_n$.

Observation (cf. Colacito, Galatos & Metcalfe 2020)

- *Gordan's theorem and Avron's theorem belong to a family of results that may be understood as "theorems of alternatives" for substructural logics.*
- *There are some general conditions for varieties of semilinear involutive commutative residuated lattices to admit a theorem of alternatives.*

Observation (cf. Colacito, Galatos & Metcalfe 2020)

- *Gordan's theorem and Avron's theorem belong to a family of results that may be understood as "theorems of alternatives" for substructural logics.*
- *There are some general conditions for varieties of semilinear involutive commutative residuated lattices to admit a theorem of alternatives.*

An **involutive commutative residuated pomonoid** is a commutative partially ordered monoid $\langle A, \leq, \cdot, 1 \rangle$ equipped with an involution $\neg$ satisfying $a \cdot b \leq c \Longleftrightarrow \neg c \cdot b \leq \neg a$ for all $a, b, c \in A$.

We define

$$a + b := \neg(\neg a \cdot \neg b), \qquad a \to b := \neg a + b, \qquad 0 := \neg 1,$$

and, inductively, for $n \in \mathbb{N}$,

$$\begin{aligned} 0a &:= 0, & a^0 &:= 1, \\ (n+1)a &:= na + a, & a^{n+1} &:= a^n \cdot a. \end{aligned}$$

If the partial order of an icr-pomonoid $\langle A, \leq, \cdot, \neg, 1 \rangle$ is a lattice order with meet and join operations $\wedge$ and $\vee$, we call the algebra $\langle A, \wedge, \vee, \cdot, \neg, 1 \rangle$ an **involutive commutative residuated lattice**.

Subdirect products of icr-lattices where the lattice order is total are called **semilinear**.

An **involutive commutative residuated pomonoid** is a commutative partially ordered monoid $\langle A, \leq, \cdot, 1\rangle$ equipped with an involution $\neg$ satisfying $a \cdot b \leq c \Longleftrightarrow \neg c \cdot b \leq \neg a$ for all $a, b, c \in A$.

We define

$$a + b := \neg(\neg a \cdot \neg b), \qquad a \to b := \neg a + b, \qquad 0 := \neg 1,$$

and, inductively, for $n \in \mathbb{N}$,

$$\begin{aligned} 0a &:= 0, & a^0 &:= 1, \\ (n+1)a &:= na + a, & a^{n+1} &:= a^n \cdot a. \end{aligned}$$

If the partial order of an icr-pomonoid $\langle A, \leq, \cdot, \neg, 1\rangle$ is a lattice order with meet and join operations $\wedge$ and $\vee$, we call the algebra $\langle A, \wedge, \vee, \cdot, \neg, 1\rangle$ an **involutive commutative residuated lattice**.

Subdirect products of icr-lattices where the lattice order is total are called **semilinear**.

An **involutive commutative residuated pomonoid** is a commutative partially ordered monoid $\langle A, \leq, \cdot, 1\rangle$ equipped with an involution $\neg$ satisfying $a \cdot b \leq c \Longleftrightarrow \neg c \cdot b \leq \neg a$ for all $a, b, c \in A$.

We define

$$a + b := \neg(\neg a \cdot \neg b), \qquad a \to b := \neg a + b, \qquad 0 := \neg 1,$$

and, inductively, for $n \in \mathbb{N}$,

$$\begin{aligned} 0a &:= 0, & a^0 &:= 1, \\ (n+1)a &:= na + a, & a^{n+1} &:= a^n \cdot a. \end{aligned}$$

If the partial order of an icr-pomonoid $\langle A, \leq, \cdot, \neg, 1\rangle$ is a lattice order with meet and join operations $\wedge$ and $\vee$, we call the algebra $\langle A, \wedge, \vee, \cdot, \neg, 1\rangle$ an **involutive commutative residuated lattice**.

Subdirect products of icr-lattices where the lattice order is total are called **semilinear**.

An **involutive commutative residuated pomonoid** is a commutative partially ordered monoid $\langle A, \leq, \cdot, 1\rangle$ equipped with an involution $\neg$ satisfying $a \cdot b \leq c \Longleftrightarrow \neg c \cdot b \leq \neg a$ for all $a, b, c \in A$.

We define

$$a + b := \neg(\neg a \cdot \neg b), \qquad a \to b := \neg a + b, \qquad 0 := \neg 1,$$

and, inductively, for $n \in \mathbb{N}$,

$$\begin{aligned} 0a &:= 0, & a^0 &:= 1, \\ (n+1)a &:= na + a, & a^{n+1} &:= a^n \cdot a. \end{aligned}$$

If the partial order of an icr-pomonoid $\langle A, \leq, \cdot, \neg, 1\rangle$ is a lattice order with meet and join operations $\wedge$ and $\vee$, we call the algebra $\langle A, \wedge, \vee, \cdot, \neg, 1\rangle$ an **involutive commutative residuated lattice**.

Subdirect products of icr-lattices where the lattice order is total are called **semilinear**.

Let $\mathbf{T}(X)$ and $\mathbf{T}_m(X)$ denote the term algebras over a set X of generators for the languages of icr-lattices and icr-pomonoids, respectively.

For any variety $\mathcal{V}$ of icr-lattices, we obtain a corresponding substructural logic by defining for any $\Sigma \cup \{s\} \subseteq T(X)$:

$$\Sigma \models_{\mathcal{V}} s \;:\Longleftrightarrow\; \text{for any homomorphism } h \text{ from } \mathbf{T}(X) \text{ to some } \mathbf{A} \in \mathcal{V}, \\ 1 \leq h(t) \text{ for all } t \in \Sigma \implies 1 \leq h(s).$$

A variety of semilinear icr-lattices $\mathcal{V}$ admits a **theorem of alternatives** if the following are equivalent for any $\Sigma \cup \{t_1, \ldots, t_n\} \subseteq T_m(X)$:

(A) $\Sigma \models_{\mathcal{V}} t_1 \vee \cdots \vee t_n$.

(B) $\Sigma \models_{\mathcal{V}} \lambda_1 t_1 + \cdots + \lambda_n t_n$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0.

Let $\mathbf{T}(X)$ and $\mathbf{T}_m(X)$ denote the term algebras over a set X of generators for the languages of icr-lattices and icr-pomonoids, respectively.

For any variety $\mathcal{V}$ of icr-lattices, we obtain a corresponding substructural logic by defining for any $\Sigma \cup \{s\} \subseteq T(X)$:

$$\begin{aligned}\Sigma \vDash_{\mathcal{V}} s \;:\Longleftrightarrow\; & \text{for any homomorphism } h \text{ from } \mathbf{T}(X) \text{ to some } \mathbf{A} \in \mathcal{V},\\ & 1 \leq h(t) \text{ for all } t \in \Sigma \implies 1 \leq h(s).\end{aligned}$$

A variety of semilinear icr-lattices $\mathcal{V}$ admits a **theorem of alternatives** if the following are equivalent for any $\Sigma \cup \{t_1, \ldots, t_n\} \subseteq T_m(X)$:

(A) $\Sigma \vDash_{\mathcal{V}} t_1 \vee \cdots \vee t_n$.

(B) $\Sigma \vDash_{\mathcal{V}} \lambda_1 t_1 + \cdots + \lambda_n t_n$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0.

Let $\mathbf{T}(X)$ and $\mathbf{T}_m(X)$ denote the term algebras over a set X of generators for the languages of icr-lattices and icr-pomonoids, respectively.

For any variety $\mathcal{V}$ of icr-lattices, we obtain a corresponding substructural logic by defining for any $\Sigma \cup \{s\} \subseteq T(X)$:

$$\Sigma \vDash_{\mathcal{V}} s \;:\Longleftrightarrow\; \text{for any homomorphism } h \text{ from } \mathbf{T}(X) \text{ to some } \mathbf{A} \in \mathcal{V}, \\ 1 \leq h(t) \text{ for all } t \in \Sigma \implies 1 \leq h(s).$$

A variety of semilinear icr-lattices $\mathcal{V}$ admits a **theorem of alternatives** if the following are equivalent for any $\Sigma \cup \{t_1, \ldots, t_n\} \subseteq T_m(X)$:

(A) $\Sigma \vDash_{\mathcal{V}} t_1 \vee \cdots \vee t_n$.

(B) $\Sigma \vDash_{\mathcal{V}} \lambda_1 t_1 + \cdots + \lambda_n t_n$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0.

Back to Gordan's Theorem:

Theorem (Colacito & Metcalfe 2019)

Let $\Sigma \cup \{t_1, \ldots, t_n\}$ be a set of group terms over the set X. The following are equivalent:

(1) $\Sigma \vDash_{\mathsf{AbLG}} t_1 \vee \cdots \vee t_n$;

(2) $\Sigma \vDash_{\mathsf{AbLG}} \lambda_1 t_1 + \cdots + \lambda_n t_n$ *for some* $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ *not all* 0;

(3) *There is no total preorder on the free abelian group over X that makes Σ positive and $\{t_1, \ldots, t_n\}$ strictly negative.*

Back to Gordan's Theorem:

Theorem (Colacito & Metcalfe 2019)

Let $\Sigma \cup \{t_1, \ldots, t_n\}$ be a set of group terms over the set X. The following are equivalent:

(1) $\Sigma \vDash_{\mathsf{AbLG}} t_1 \vee \cdots \vee t_n$;

(2) $\Sigma \vDash_{\mathsf{AbLG}} \lambda_1 t_1 + \cdots + \lambda_n t_n$ *for some* $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ *not all* 0;

(3) *There is no total preorder on the free abelian group over X that makes Σ positive and $\{t_1, \ldots, t_n\}$ strictly negative.*

A **preorder** $\preceq$ on an icr-pomonoid $\mathbf{A}$ is a reflexive and transitive binary relation on A that

(i) extends the partial order $\leq$, i.e., $\leq \subseteq \preceq$;

(ii) $a \preceq b$ implies $ca \preceq cb$ for all $a, b, c \in A$;

(iii) $a \preceq b$ implies $\neg b \preceq \neg a$ for all $a \in A$.

We call $\preceq$ **total** if $a \preceq b$ or $b \preceq a$ for all $a, b \in A$.

A **preorder** $\preceq$ on an icr-pomonoid $\mathbf{A}$ is a reflexive and transitive binary relation on A that

(i) extends the partial order $\leq$, i.e., $\leq \subseteq \preceq$;

(ii) $a \preceq b$ implies $ca \preceq cb$ for all $a, b, c \in A$;

(iii) $a \preceq b$ implies $\neg b \preceq \neg a$ for all $a \in A$.

We call $\preceq$ **total** if $a \preceq b$ or $b \preceq a$ for all $a, b \in A$.

A **preorder** $\preceq$ on an icr-pomonoid $\mathbf{A}$ is a reflexive and transitive binary relation on A that

(i) extends the partial order $\leq$, i.e., $\leq \subseteq \preceq$;

(ii) $a \preceq b$ implies $ca \preceq cb$ for all $a, b, c \in A$;

(iii) $a \preceq b$ implies $\neg b \preceq \neg a$ for all $a \in A$.

We call $\preceq$ **total** if $a \preceq b$ or $b \preceq a$ for all $a, b \in A$.

A **preorder** $\preceq$ on an icr-pomonoid $\mathbf{A}$ is a reflexive and transitive binary relation on A that

(i) extends the partial order $\leq$, i.e., $\leq \subseteq \preceq$;

(ii) $a \preceq b$ implies $ca \preceq cb$ for all $a, b, c \in A$;

(iii) $a \preceq b$ implies $\neg b \preceq \neg a$ for all $a \in A$.

We call $\preceq$ **total** if $a \preceq b$ or $b \preceq a$ for all $a, b \in A$.

A **preorder** $\preceq$ on an icr-pomonoid $\mathbf{A}$ is a reflexive and transitive binary relation on A that

(i) extends the partial order $\leq$, i.e., $\leq \subseteq \preceq$;

(ii) $a \preceq b$ implies $ca \preceq cb$ for all $a, b, c \in A$;

(iii) $a \preceq b$ implies $\neg b \preceq \neg a$ for all $a \in A$.

We call $\preceq$ **total** if $a \preceq b$ or $b \preceq a$ for all $a, b \in A$.

- If $\preceq$ is a preorder on an icr-pomonoid $\mathbf{A}$, define its positive cone to be

$$P := \{a \in A \mid 1 \preceq a\}.$$

- The positive cone P is a filter submonoid of $\mathbf{A}$, i.e.,

$$P \text{ is an upset of } \langle A, \leq \rangle, 1 \in P, \text{ and } a, b \in P \text{ implies } ab \in P.$$

- For any filter submonoid $P \subseteq A$, the binary relation $\preceq_P$ on A defined by

$$a \preceq_P b \iff a \to b \in P$$

is a preorder on $\mathbf{A}$ with positive cone P.

- We can study preorders via filter submonoids.

- If $\preceq$ is a preorder on an icr-pomonoid $\mathbf{A}$, define its positive cone to be

$$P := \{a \in A \mid 1 \preceq a\}.$$

- The positive cone P is a filter submonoid of $\mathbf{A}$, i.e.,

$$P \text{ is an upset of } \langle A, \leq \rangle, 1 \in P, \text{ and } a, b \in P \text{ implies } ab \in P.$$

- For any filter submonoid $P \subseteq A$, the binary relation $\preceq_P$ on A defined by

$$a \preceq_P b \iff a \to b \in P$$

is a preorder on $\mathbf{A}$ with positive cone P.

- We can study preorders via filter submonoids.

- If $\preceq$ is a preorder on an icr-pomonoid $\mathbf{A}$, define its positive cone to be

$$P := \{a \in A \mid 1 \preceq a\}.$$

- The positive cone P is a filter submonoid of $\mathbf{A}$, i.e.,

$$P \text{ is an upset of } \langle A, \leq \rangle, 1 \in P, \text{ and } a, b \in P \text{ implies } ab \in P.$$

- For any filter submonoid $P \subseteq A$, the binary relation $\preceq_P$ on A defined by

$$a \preceq_P b \iff a \to b \in P$$

is a preorder on $\mathbf{A}$ with positive cone P.

- We can study preorders via filter submonoids.

- If $\preceq$ is a preorder on an icr-pomonoid $\mathbf{A}$, define its positive cone to be

$$P := \{a \in A \mid 1 \preceq a\}.$$

- The positive cone P is a filter submonoid of $\mathbf{A}$, i.e.,

$$P \text{ is an upset of } \langle A, \leq \rangle, 1 \in P, \text{ and } a, b \in P \text{ implies } ab \in P.$$

- For any filter submonoid $P \subseteq A$, the binary relation $\preceq_P$ on A defined by

$$a \preceq_P b \iff a \to b \in P$$

is a preorder on $\mathbf{A}$ with positive cone P.

- We can study preorders via filter submonoids.

Let $\mathcal{V}$ be any variety of semilinear icr-lattices and let $\mathbf{F}(X)$ denote the free algebra of $\mathcal{V}$ over a set X. Let $\mathbf{F}_m(X)$ be the icr-pomonoid formed by the $\{\cdot, \neg, 1\}$-subreduct of $\mathbf{F}(X)$ generated by X, partially ordered by the restriction of the lattice order of $\mathbf{F}(X)$.

Lemma

The following are equivalent for any $\Sigma \cup \{t_1, \ldots, t_n\} \subseteq T_m(X)$:

(1) *There is no total preorder on $\mathbf{F}_m(X)$ containing Σ and omitting $t_1, \ldots, t_n$.*

(2) $\Sigma \vDash_{\mathcal{V}} t_1 \vee \cdots \vee t_n$.

Let $\mathcal{V}$ be any variety of semilinear icr-lattices and let $\mathbf{F}(X)$ denote the free algebra of $\mathcal{V}$ over a set X. Let $\mathbf{F}_m(X)$ be the icr-pomonoid formed by the $\{\cdot, \neg, 1\}$-subreduct of $\mathbf{F}(X)$ generated by X, partially ordered by the restriction of the lattice order of $\mathbf{F}(X)$.

Lemma

The following are equivalent for any $\Sigma \cup \{t_1, \ldots, t_n\} \subseteq T_m(X)$:

(1) *There is no total preorder on $\mathbf{F}_m(X)$ containing Σ and omitting $t_1, \ldots, t_n$.*

(2) $\Sigma \vDash_{\mathcal{V}} t_1 \vee \cdots \vee t_n$.

Let $\mathcal{V}$ be any variety of semilinear icr-lattices and let $\mathbf{F}(X)$ denote the free algebra of $\mathcal{V}$ over a set X. Let $\mathbf{F}_m(X)$ be the icr-pomonoid formed by the $\{\cdot, \neg, 1\}$-subreduct of $\mathbf{F}(X)$ generated by X, partially ordered by the restriction of the lattice order of $\mathbf{F}(X)$.

Lemma

The following are equivalent for any $\Sigma \cup \{t_1, \ldots, t_n\} \subseteq T_m(X)$:

(1) *There is no total preorder on $\mathbf{F}_m(X)$ containing Σ and omitting $t_1, \ldots, t_n$.*

(2) $\Sigma \models_{\mathcal{V}} t_1 \vee \cdots \vee t_n$.

Theorem

Let $\mathcal{V}$ be a variety of semilinear icr-lattices satisfying $0 \leq 1$, $1 \leq x \vee \neg x$, and for each $n \in \mathbb{N}$, the equation $(nx)^k \leq m(x^n)$ for some $m \in \mathbb{N}^+, k \in \mathbb{N}$. Then the following are equivalent for any $\Sigma \cup \{t_1, \ldots, t_n\} \subseteq T_m(X)$:

(1) *There is no total preorder on $\mathbf{F}_m(X)$ containing Σ and omitting $t_1, \ldots, t_n$.*

(2) $\Sigma \vDash_{\mathcal{V}} \lambda_1 t_1 + \ldots + \lambda_n t_n$ *for some $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0.*

This result applies to the varieties of abelian ℓ-groups, Sugihara monoids, and semilinear icr-lattices satisfying $x^n \approx nx$ for each $n \in \mathbb{N}^+$.

Theorem

Let $\mathcal{V}$ be a variety of semilinear icr-lattices satisfying $0 \leq 1$, $1 \leq x \vee \neg x$, and for each $n \in \mathbb{N}$, the equation $(nx)^k \leq m(x^n)$ for some $m \in \mathbb{N}^+, k \in \mathbb{N}$. Then the following are equivalent for any $\Sigma \cup \{t_1, \ldots, t_n\} \subseteq T_m(X)$:

(1) *There is no total preorder on $\mathbf{F}_m(X)$ containing Σ and omitting $t_1, \ldots, t_n$.*

(2) $\Sigma \vDash_{\mathcal{V}} \lambda_1 t_1 + \ldots + \lambda_n t_n$ *for some $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0.*

This result applies to the varieties of abelian ℓ-groups, Sugihara monoids, and semilinear icr-lattices satisfying $x^n \approx nx$ for each $n \in \mathbb{N}^+$.

Proof sketch:

- Let P_0 be the filter submonoid generated by $\{t \in F_m(X) \mid 1 \leq t\} \cup \Sigma$.
- Let
$$S := \{\lambda_1 t_1 + \ldots + \lambda_n t_n \mid \{\lambda_1, \ldots, \lambda_n\} \in \mathbb{N} \text{ not all } 0 \}.$$
- Suppose that $\Sigma \not\models_{\mathcal{V}} \lambda_1 t_1 + \cdots + \lambda_n t_n$ for every $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0. This is equivalent to assuming $S \cap P_0 = \varnothing$
- Enumerate the elements in $F_m(X)$ as $(t_i)_{i \in \mathbb{N}}$ and define
$$P_{i+1} := \begin{cases} \langle P_i \cup \{t_i\} \rangle & \text{if } \langle P_i \cup \{t_i\} \rangle \cap S = \varnothing, \\ \langle P_i \cup \{\neg t_i\} \rangle & \text{otherwise.} \end{cases}$$

Lemma

Let P be a preorder on an icr-pomonoid $\mathbf{A}$ and $a \in A$. Then the preorder generated by $P \cup \{a\}$ is $\{c \in A \mid a^n \to c \in P$ for some $n \in \mathbb{N}\}$.

- $P := \cup_{N \in \mathbb{N}} P_N$ satisfies $S \cap P = \varnothing$, it is a total preorder on $\mathbf{F}_m(X)$, contains Σ and omits $t_1, \ldots, t_n$.

Proof sketch:

- Let P_0 be the filter submonoid generated by $\{t \in F_m(X) \mid 1 \leq t\} \cup \Sigma$.
- Let

$$S := \{\lambda_1 t_1 + \ldots + \lambda_n t_n \mid \{\lambda_1, \ldots, \lambda_n\} \in \mathbb{N} \text{ not all } 0 \}.$$

- Suppose that $\Sigma \not\models_{\mathcal{V}} \lambda_1 t_1 + \cdots + \lambda_n t_n$ for every $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0. This is equivalent to assuming $S \cap P_0 = \varnothing$
- Enumerate the elements in $F_m(X)$ as $(t_i)_{i \in \mathbb{N}}$ and define

$$P_{i+1} := \begin{cases} \langle P_i \cup \{t_i\} \rangle & \text{if } \langle P_i \cup \{t_i\} \rangle \cap S = \varnothing, \\ \langle P_i \cup \{\neg t_i\} \rangle & \text{otherwise.} \end{cases}$$

Lemma

Let P be a preorder on an icr-pomonoid $\mathbf{A}$ and $a \in A$. Then the preorder generated by $P \cup \{a\}$ is $\{c \in A \mid a^n \to c \in P$ for some $n \in \mathbb{N}\}$.

- $P := \cup_{N \in \mathbb{N}} P_N$ satisfies $S \cap P = \varnothing$, it is a total preorder on $\mathbf{F}_m(X)$, contains Σ and omits $t_1, \ldots, t_n$.

Proof sketch:

- Let P_0 be the filter submonoid generated by $\{t \in F_m(X) \mid 1 \leq t\} \cup \Sigma$.
- Let

$$S := \{\lambda_1 t_1 + \ldots + \lambda_n t_n \mid \{\lambda_1, \ldots, \lambda_n\} \in \mathbb{N} \text{ not all 0 } \}.$$

- Suppose that $\Sigma \not\models_{\mathcal{V}} \lambda_1 t_1 + \cdots + \lambda_n t_n$ for every $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0. This is equivalent to assuming $S \cap P_0 = \varnothing$
- Enumerate the elements in $F_m(X)$ as $(t_i)_{i \in \mathbb{N}}$ and define

$$P_{i+1} := \begin{cases} \langle P_i \cup \{t_i\} \rangle & \text{if } \langle P_i \cup \{t_i\} \rangle \cap S = \varnothing, \\ \langle P_i \cup \{\neg t_i\} \rangle & \text{otherwise.} \end{cases}$$

Lemma

Let P be a preorder on an icr-pomonoid $\mathbf{A}$ and $a \in A$. Then the preorder generated by $P \cup \{a\}$ is $\{c \in A \mid a^n \to c \in P \text{ for some } n \in \mathbb{N}\}$.

- $P := \cup_{N \in \mathbb{N}} P_N$ satisfies $S \cap P = \varnothing$, it is a total preorder on $\mathbf{F}_m(X)$, contains Σ and omits $t_1, \ldots, t_n$.

Proof sketch:

- Let P_0 be the filter submonoid generated by $\{t \in F_m(X) \mid 1 \leq t\} \cup \Sigma$.
- Let
$$S := \{\lambda_1 t_1 + \ldots + \lambda_n t_n \mid \{\lambda_1, \ldots, \lambda_n\} \in \mathbb{N} \text{ not all } 0 \}.$$
- Suppose that $\Sigma \not\models_{\mathcal{V}} \lambda_1 t_1 + \cdots + \lambda_n t_n$ for every $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0. This is equivalent to assuming $S \cap P_0 = \varnothing$
- Enumerate the elements in $F_m(X)$ as $(t_i)_{i\in\mathbb{N}}$ and define
$$P_{i+1} := \begin{cases} \langle P_i \cup \{t_i\}\rangle & \text{if } \langle P_i \cup \{t_i\}\rangle \cap S = \varnothing, \\ \langle P_i \cup \{\neg t_i\}\rangle & \text{otherwise.} \end{cases}$$

Lemma

Let P be a preorder on an icr-pomonoid $\mathbf{A}$ and $a \in A$. Then the preorder generated by $P \cup \{a\}$ is $\{c \in A \mid a^n \to c \in P$ for some $n \in \mathbb{N}\}$.

- $P := \cup_{N\in\mathbb{N}} P_N$ satisfies $S \cap P = \varnothing$, it is a total preorder on $\mathbf{F}_m(X)$, contains Σ and omits $t_1, \ldots, t_n$.

Proof sketch:

- Let P_0 be the filter submonoid generated by $\{t \in F_m(X) \mid 1 \leq t\} \cup \Sigma$.
- Let
$$S := \{\lambda_1 t_1 + \ldots + \lambda_n t_n \mid \{\lambda_1, \ldots, \lambda_n\} \in \mathbb{N} \text{ not all } 0\ \}.$$
- Suppose that $\Sigma \not\models_{\mathcal{V}} \lambda_1 t_1 + \cdots + \lambda_n t_n$ for every $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0. This is equivalent to assuming $S \cap P_0 = \varnothing$
- Enumerate the elements in $F_m(X)$ as $(t_i)_{i \in \mathbb{N}}$ and define
$$P_{i+1} := \begin{cases} \langle P_i \cup \{t_i\}\rangle & \text{if } \langle P_i \cup \{t_i\}\rangle \cap S = \varnothing, \\ \langle P_i \cup \{\neg t_i\}\rangle & \text{otherwise.} \end{cases}$$

Lemma

Let P be a preorder on an icr-pomonoid $\mathbf{A}$ and $a \in A$. Then the preorder generated by $P \cup \{a\}$ is $\{c \in A \mid a^n \to c \in P$ for some $n \in \mathbb{N}\}$.

- $P := \cup_{N \in \mathbb{N}} P_N$ satisfies $S \cap P = \varnothing$, it is a total preorder on $\mathbf{F}_m(X)$, contains Σ and omits $t_1, \ldots, t_n$.

Proposition

Let $\mathcal{V}$ be a variety of semilinear icr-lattices $\mathcal{V}$ that admits a theorem of alternatives. Then $\mathcal{V}$ satisfies $0 \leq 1$, $1 \leq x \vee \neg x$, and for each $n \in \mathbb{N}$, the equation $(nx)^k \leq m(x^n)$ for some $m \in \mathbb{N}^+, k \in \mathbb{N}$.

- $1 \leq x \to x = \neg x + x$, so by the theorem of alternatives we obtain that

$$1 \leq x \vee \neg x.$$

- From $1 \leq (\neg x)^n \vee x^n$, using the theorem of alternatives, we get $1 \leq k(\neg x)^n + m(x^n)$ for some $k, m \in \mathbb{N}$ not both 0. Then we have that

$$\begin{aligned} 1 \leq k(\neg x)^n + m(x^n) &\iff 1 \leq \neg[(nx)^k] + m(x^n) \\ &\iff 1 \leq (nx)^k \to m(x^n) \\ &\iff (nx)^k \leq m(x^n) \end{aligned}$$

- $1 \leq x \to x = \neg x + x$, so by the theorem of alternatives we obtain that

$$1 \leq x \vee \neg x.$$

- From $1 \leq (\neg x)^n \vee x^n$, using the theorem of alternatives, we get $1 \leq k(\neg x)^n + m(x^n)$ for some $k, m \in \mathbb{N}$ not both 0. Then we have that

$$\begin{aligned} 1 \leq k(\neg x)^n + m(x^n) &\iff 1 \leq \neg[(nx)^k] + m(x^n) \\ &\iff 1 \leq (nx)^k \to m(x^n) \\ &\iff (nx)^k \leq m(x^n) \end{aligned}$$

- $1 \leq x \to x = \neg x + x$, so by the theorem of alternatives we obtain that

$$1 \leq x \vee \neg x.$$

- From $1 \leq (\neg x)^n \vee x^n$, using the theorem of alternatives, we get $1 \leq k(\neg x)^n + m(x^n)$ for some $k, m \in \mathbb{N}$ not both 0. Then we have that

$$\begin{aligned} 1 \leq k(\neg x)^n + m(x^n) &\iff 1 \leq \neg[(nx)^k] + m(x^n) \\ &\iff 1 \leq (nx)^k \to m(x^n) \\ &\iff (nx)^k \leq m(x^n) \end{aligned}$$

- $1 \leq x \to x = \neg x + x$, so by the theorem of alternatives we obtain that

$$1 \leq x \vee \neg x.$$

- From $1 \leq (\neg x)^n \vee x^n$, using the theorem of alternatives, we get $1 \leq k(\neg x)^n + m(x^n)$ for some $k, m \in \mathbb{N}$ not both 0. Then we have that

$$\begin{aligned} 1 \leq k(\neg x)^n + m(x^n) &\iff 1 \leq \neg[(nx)^k] + m(x^n) \\ &\iff 1 \leq (nx)^k \to m(x^n) \\ &\iff (nx)^k \leq m(x^n) \end{aligned}$$

Theorem

A variety of semilinear icr-lattices $\mathcal{V}$ admits a theorem of alternatives if, and only if, it satisfies $0 \leq 1$, $1 \leq x \vee \neg x$, and for each $n \in \mathbb{N}$, the equation $(nx)^k \leq m(x^n)$ for some $m \in \mathbb{N}^+, k \in \mathbb{N}$.

Theorem (Colacito & Metcalfe 2019)

Let $\{t_1, \ldots, t_n\}$ *be a set of group terms over the set* X*. The following are equivalent:*

(1) AbLG $\models t_1 \vee \cdots \vee t_n$*;*

(2) AbLG $\models \lambda_1 t_1 + \cdots + \lambda_n t_n$ *for some* $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ *not all* 0*;*

Theorem (Colacito & Metcalfe 2019)

Let $\{t_1, \ldots, t_n\}$ be a set of group terms over the set X. The following are equivalent:

(1) $\mathsf{AbLG} \vDash t_1 \vee \cdots \vee t_n$;

(2) $\mathsf{AbLG} \vDash \lambda_1 t_1 + \cdots + \lambda_n t_n$ *for some* $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ *not all* 0;

(3) *There is no order on the free abelian group over* X *containing* $\{t_1, \ldots, t_n\}$.

- The Validity Problem for AbLG 'reduces to' the Order Problem for free Abelian groups

Theorem (Colacito & Metcalfe 2019)

Let $\{t_1, \ldots, t_n\}$ *be a set of group terms over the set* X*. The following are equivalent:*

(1) $\mathsf{AbLG} \models t_1 \vee \cdots \vee t_n$*;*

(2) $\mathsf{AbLG} \models \lambda_1 t_1 + \cdots + \lambda_n t_n$ *for some* $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ *not all* 0*;*

(3) *There is no order on the free abelian group over* X *containing* $\{t_1, \ldots, t_n\}$*.*

(4) *The subsemigroup of the free abelian group over* X *generated by* $\{t_1, \ldots, t_n\}$ *contains* 1*.*

- The Validity Problem for AbLG ‘reduces to’ the Order Problem for free Abelian groups and the Identity Problem for free Abelian groups.

Theorem (Colacito & Metcalfe 2019)

Let $\{t_1, \ldots, t_n\}$ be a set of group terms over the set X. The following are equivalent:

(1) AbLG $\vDash t_1 \vee \cdots \vee t_n$*;*

(2) AbLG $\vDash \lambda_1 t_1 + \cdots + \lambda_n t_n$ *for some* $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ *not all* 0*;*

(3) *There is no order on the free abelian group over X containing $\{t_1, \ldots, t_n\}$.*

(4) *The subsemigroup of the free abelian group over X generated by $\{t_1, \ldots, t_n\}$ contains 1.*

- The Validity Problem for AbLG 'reduces to' the Order Problem for free Abelian groups and the Identity Problem for free Abelian groups.
- Can we say something similar for the substructural case?

Thank you!