Varieties of distributive ℓ-pregroups

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Lattice-ordered groups (ℓ-groups)

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- Involutive residuated lattices which generalize ℓ-groups
- **Distributive ℓ-pregroups:** closer to ℓ-groups (distrib. lattice reduct)
- **Open Problem:** Is every ℓ-pregroup distributive?

Lattice-ordered pregroups

An **ℓ-pregroup** is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, ^{\ell}, ^{r}, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid s.t. $\cdot$ is order-preserving,
- and for every $a \in L$, $a^{\ell}a \leq 1 \leq aa^{\ell}$ and $aa^{r} \leq 1 \leq a^{r}a$.

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Any ℓ-group is a distributive ℓ-pregroup with $a^{\ell} = a^{r} = a^{-1}$

Periodic ℓ-pregroups

An ℓ-pregroup $\mathbf{L}$ is called ***n*-periodic** (for $n \geq 1$) if it satisfies

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- $\mathsf{DLP} = \bigvee_n \mathsf{LP}_n$ (Galatos–Gallardo 2025)

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Approach: Representation Thm. for distributive ℓ-pregroups as endomorphism ℓ-pregroups of chains

Endomorphism ℓ-pregroup of a chain

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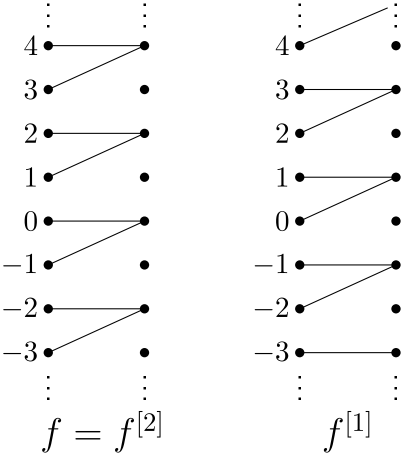

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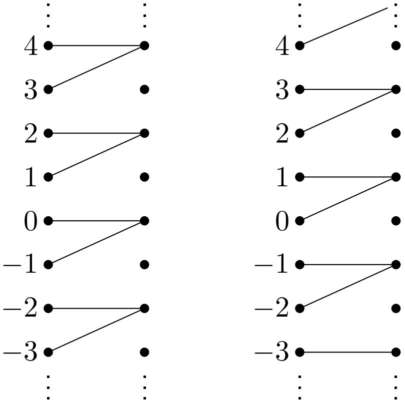

$f = f^{[2]}$ $\qquad$ $f^{[1]}$

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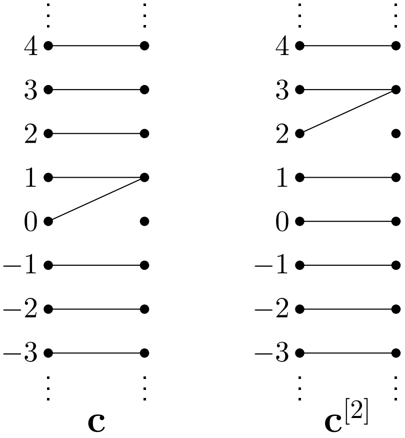

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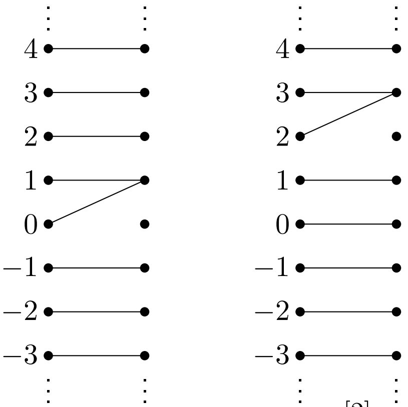

$\mathbf{c}$ $\qquad$ $\mathbf{c}^{[2]}$

- For $\mathbf{F}_{\mathrm{fs}}(\mathbb{Z}) := \mathbf{Sg}(\mathbf{c}) \leq \mathbf{F}(\mathbb{Z})$, $\mathbb{V}(\mathbf{F}_{\mathrm{fs}}(\mathbb{Z})) = \mathrm{DLP}$ (Galatos–Gallardo 2024)

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- Axiomatization of $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ for every $n \geq 1$ (Galatos–S. 2026)

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$\Longrightarrow \mathbb{V}(\mathbf{Sg}(f)) =$ DLP

Special case $\mathrm{F}(\mathbb{Z})$

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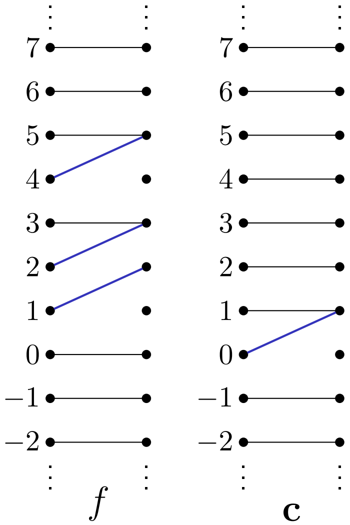

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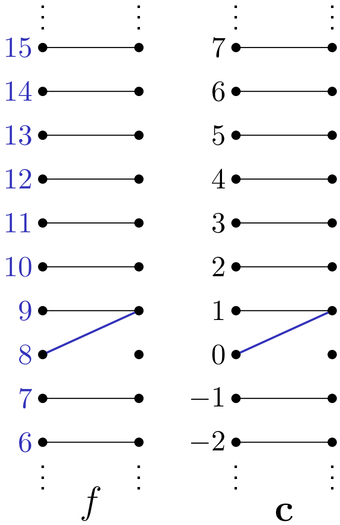

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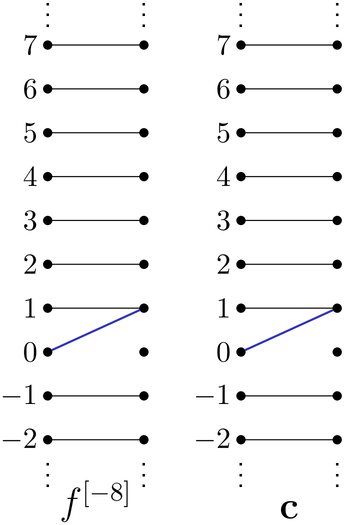

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If $f \in \mathrm{F}(\mathbb{Z})$ is non-periodic, then one of the following holds

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Special case $\mathbf{F}(\mathbb{Z})$: reduction to idempotents

Periodicity of $a \in L$: $\mathrm{per}(a) := \inf\{k \geq 1 : a^{[k]} = a\}$

Lemma (Galatos–S. 2026+)

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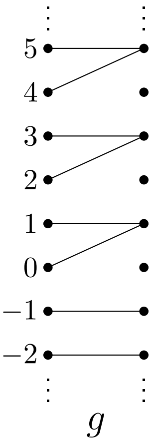

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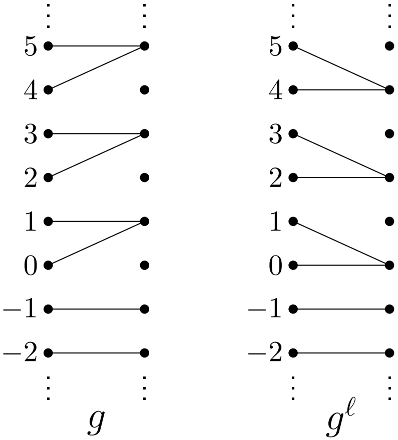

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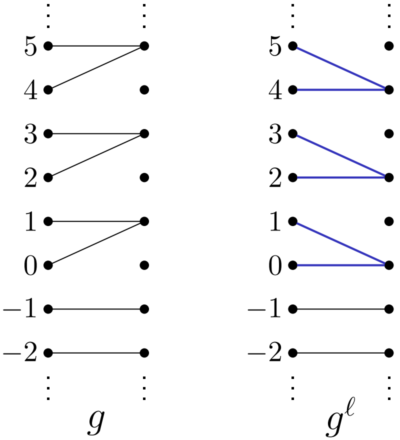

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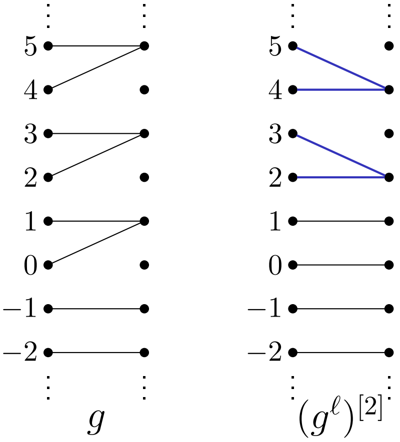

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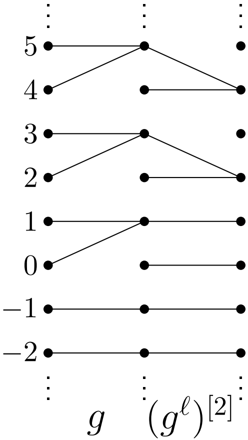

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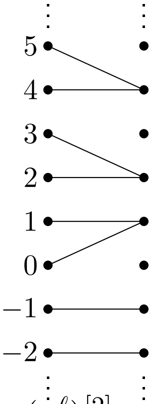

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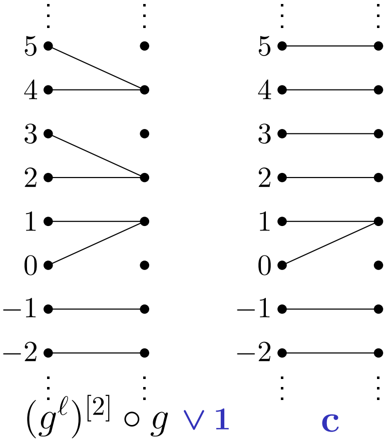

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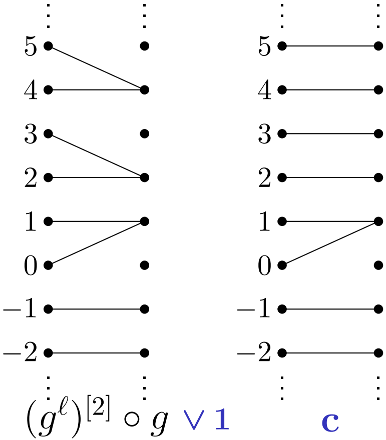

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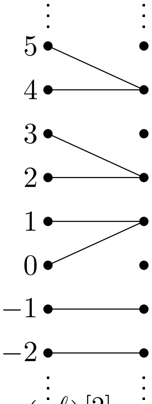

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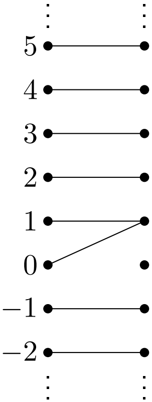

$\mathbf{c}$

General case $\mathbf{F}(\mathbf{J} \overrightarrow{\times} \mathbb{Z})$

General case: reduction to idempotents

Lemma (Galatos–S. 2026+)

If $f \in \mathrm{F}(\mathbf{J} \overrightarrow{\times} \mathbb{Z})$ is non-periodic, one of the following holds

(1) $\mathbf{Sg}(f)$ contains a non-periodic positive idempotent

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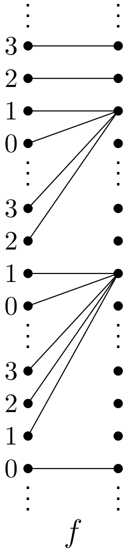

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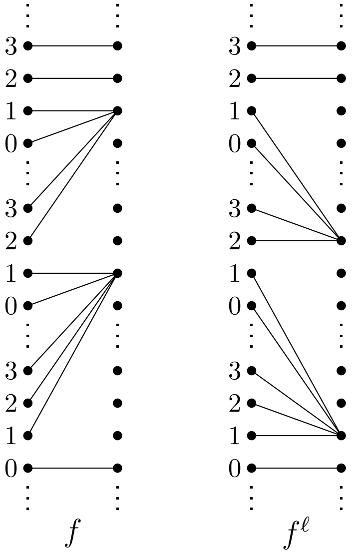

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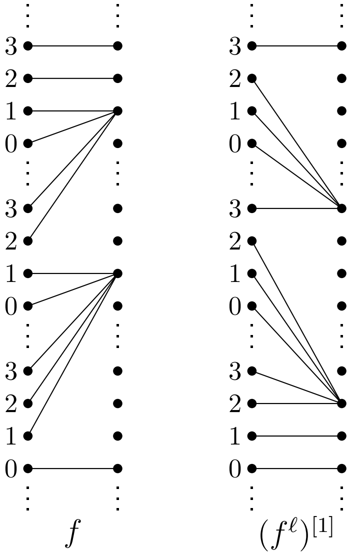

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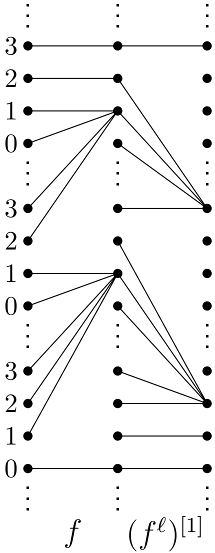

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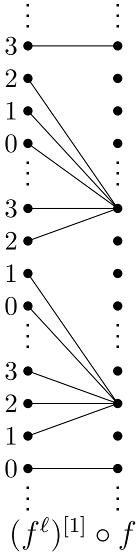

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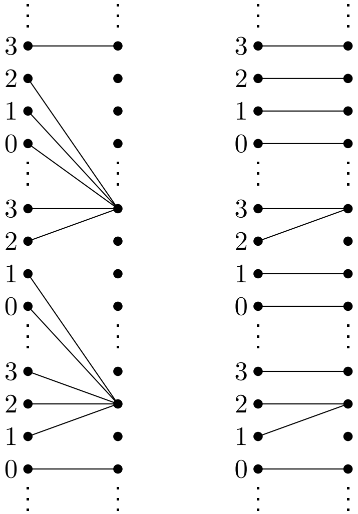

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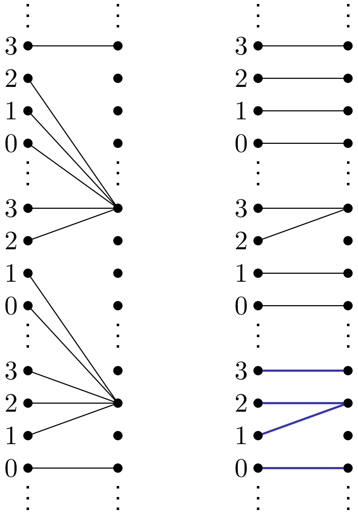

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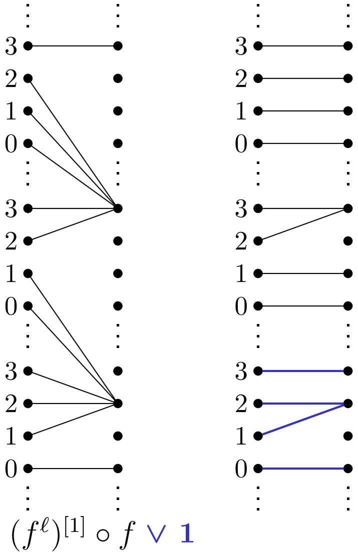

Conclusion

Theorem (Galatos–S. 2026+)

Every proper subvariety V of DLP is periodic, i.e., $\exists n \geq 1$, V $\subseteq$ LP$_n$

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The subvariety lattice of DLP decomposes into disjoint intervals: $[\bigvee_{q\in Q}\mathbb{V}(\mathbf{F}_q(\mathbb{Z})), \mathsf{LP}_n]$, Q contains maximal prime powers of n.

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Thank You!

References

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[2] Nikolaos Galatos and Isis A. Gallardo. *Generation and decidability for periodic ℓ-pregroups*. J. Algebra 669 (2025), 301–340.

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