A Nested Approach to Relevant Logic

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The basic relevant logic B^+

- B^+ acts as the foundational positive logic within the relevant logic framework [Routley and Meyer, 1973]

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  - logic CB by adding a Boolean negation ($\neg$)
  - some relevant modal logics such as $B^+K^\Box$
- **Semantics:** Interpreted via Routley-Meyer frames containing a *ternary accessibility relation*
  - in standard relational semantics, accessibility relations are binary
  - for relevant implication $\rightarrow$, a ternary relation $Rxyz$ is required

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- **nested sequent**:

$S = A_1, \ldots, A_k \Rightarrow B_1, \ldots, B_j, [\,T_1\,], \ldots, [\,T_m\,]$ [Brünnler, 2009]

$$tr(S):\quad \begin{tikzcd}[column sep=small] tr(T_1) & & \cdots & & tr(T_m) \\ & & A_1, \ldots, A_k \Rightarrow B_1, \ldots, B_k \arrow[ull, "{[\cdot]}"] \arrow[urr, "{[\cdot]}"'] & & \end{tikzcd}$$

where each T_i is nested sequent

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Proof theory for relevant logic

- Existing calculi for B^+/B are labeled or display calculi provided in [Belnap, 1982, Kurokawa and Negri, 2020, De Martin Polo, 2023]

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In this work...

- we developed the first **nested sequent calculus** for B^+ which directly internalizes ternary relations
- we extended technique from intuitionistic modal logic [Straßburger, 2013, Gao and Olivetti, 2026]

Routley-Meyer Relational Semantics

The **language** Frm is generated by:

$$\mathrm{Frm} \ni A := p \in \mathrm{At} \mid (A \wedge A) \mid (A \vee A) \mid (A \to A)$$

where At is a countable set of propositional atoms.

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A B^+**-frame** is a quadruple $\mathscr{F} = (\mathscr{O}, K, R, \leq)$ where

(i) K is a non-empty set of worlds

(ii) $\mathscr{O} \subseteq K$ is the set of *regular worlds*

(iii) $R \subseteq K \times K \times K$ is a ternary accessibility relation

(iv) $\leq$ is a partial order on K, defined as $a \leq b \iff \exists u \in \mathscr{O}\ \&\ Ruab$

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Moreover, for all $a, b, c, d \in K$:

- **(F1)** $a \leq a$
- **(F2)** If $a \leq b$ and $Rbcd$, then $Racd$.

Routley-Meyer Relational Semantics

A B^+**-model** $\mathscr{M} = (\mathscr{F}, V)$ pairs a frame with a valuation $V : \mathrm{At} \to \mathscr{P}(K)$ which is monotonic:

$$\text{if } x \in V(p) \text{ and } x \leq y, \text{ then } y \in V(p)$$

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Forcing Relation $\Vdash$

- $\mathcal{M}, x \Vdash p \iff x \in V(p)$
- $\mathcal{M}, x \Vdash B \wedge C \iff \mathcal{M}, x \Vdash B$ and $\mathcal{M}, x \Vdash C$
- $\mathcal{M}, x \Vdash B \vee C \iff \mathcal{M}, x \Vdash B$ or $\mathcal{M}, x \Vdash C$
- $\mathcal{M}, x \Vdash B \to C \iff \forall y, z \in K(Rxyz \mathbin{\&} \mathcal{M}, y \Vdash B)$ then $\mathcal{M}, z \Vdash C$

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Monotonicity property

For any formula $A \in \mathrm{Frm}$, if $\mathscr{M}, x \Vdash A$ and $x \leq y$, then $\mathscr{M}, y \Vdash A$.

The Hilbert system $\mathscr{H}_{B+}$ [Routley and Meyer, 1973]

ax1 $A \to A$

ax2 $(A \to B) \to ((B \to C) \to (A \to C))$

ax3 $(A \to (A \to B)) \to (A \to B)$

ax4 $(A_1 \wedge A_2) \to A_i$

ax5 $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

ax6 $A_i \to (A_1 \vee A_2)$

ax7 $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

ax8 $(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$

rules $\dfrac{A \quad A\to B}{B}$ $\qquad \dfrac{A \quad B}{A \wedge B}$ $\qquad \dfrac{A\to B \quad C\to D}{(B\to C)\to(A\to D)}$

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Theorem (Soundness and completeness)

A is derivable in $\mathscr{H}_{\mathrm{B}^+}$ if and only if A is B^+-valid.

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- However, for a ternary relation *Rxyz*, a world *x* sees a *pair* of worlds (y, z).

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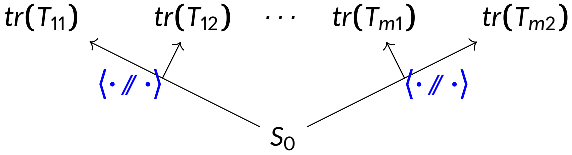
- However, for a ternary relation $Rxyz$, a world x sees a *pair* of worlds (y, z).
- **Our solution:** Using binary nested blocks $\langle \cdot \mathbin{/\!\!/} \cdot \rangle$

$tr(S)$:



Syntax of nested sequents

Formulas are polarized by $\bullet$ (left/input) and $\circ$ (right/output)

A full *nested sequent* is generated by the grammar:

$$\Gamma :: \Lambda, \Pi$$

where

- left-sided $\Lambda :: \varnothing \mid A^{\bullet} \mid \langle \Lambda \mathbin{/\!\!/} \Lambda \rangle \mid \Lambda, \Lambda$
- right-sided $\Pi :: A^{\circ} \mid \langle \Lambda \mathbin{/\!\!/} \Gamma \rangle \mid \langle \Gamma \mathbin{/\!\!/} \Lambda \rangle$

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Example

$\Sigma = p \vee q^\bullet, \langle p^\bullet \mathbin{/\!\!/} p \vee q^\bullet \rangle, \langle q^\circ \mathbin{/\!\!/} p \rightarrow q^\bullet \rangle$ is a full nested sequent,
$\Lambda = p \vee q^\bullet, \langle p^\bullet \mathbin{/\!\!/} p \vee q^\bullet \rangle, \langle p^\bullet, q^\bullet \mathbin{/\!\!/} p \rightarrow q^\bullet \rangle$ is a left-sided sequent.

Basic notions with nesting

- **context:** $G\{\ \}$ denotes a nested sequent holding an empty component with the unique palceholder $\{\ \}$ and $G\{\Sigma\}$ refers to a nested sequent which is fulfilled with Σ to $G\{\ \}$.
- **pruning:** Σ^* means to remove the unique output formula. By $G^*\{\ \}$ we mean removing the unique output formula in G.
- distinction between regular (non-nesting) and irregular (arbitrary nesting) worlds:

$$\overbrace{\Sigma}^{x\in\mathcal{O}}, \langle \overbrace{\Phi_1}^{a\in K} /\!\!/ \overbrace{\Phi_2}^{b\in K} \rangle \quad \text{and} \quad G\{\overbrace{\Sigma}^{a}, \langle \overbrace{\Phi_1}^{b} /\!\!/ \overbrace{\Phi_2}^{c} \rangle\}$$

Example

$$\Phi = r^{\bullet}, p \vee q^{\bullet}, \langle p^{\bullet} \mathbin{/\!\!/} r \rightarrow s^{\bullet} \rangle, \langle q^{\circ} \mathbin{/\!\!/} p \rightarrow q^{\bullet} \rangle$$

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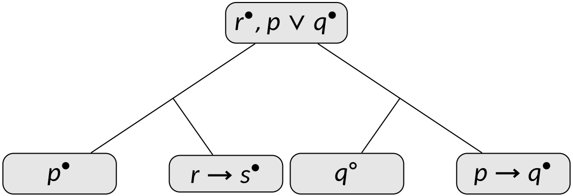

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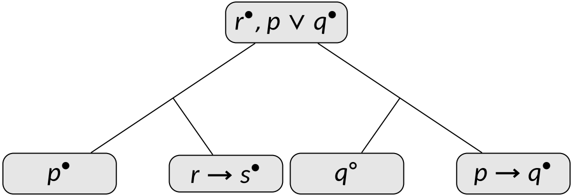

- $\Phi^* = r^\bullet, p \vee q^\bullet, \langle p^\bullet \mathbin{/\!\!/} r \rightarrow s^\bullet \rangle, \langle - \mathbin{/\!\!/} p \rightarrow q^\bullet \rangle$

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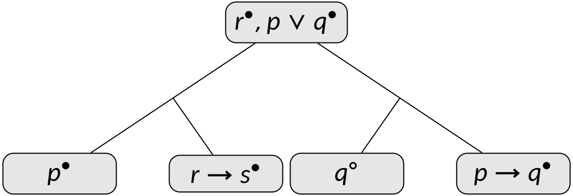

- $\Phi^* = r^\bullet, p \vee q^\bullet, \langle p^\bullet \mathbin{/\!\!/} r \to s^\bullet \rangle, \langle - \mathbin{/\!\!/} p \to q^\bullet \rangle$
- $\Phi = G\{r \to s^\bullet\}$ where $G\{\ \} = r^\bullet, p \vee q^\bullet, \langle p^\bullet \mathbin{/\!\!/} \{\ \} \rangle, \langle q^\circ \mathbin{/\!\!/} p \to q^\bullet \rangle$

The nested calculus $\mathbf{C}_{\mathrm{B}^+}$: basic rules

Initial sequents:

$$\dfrac{}{G\{\Sigma, p^{\bullet}, p^{\circ}\}}\ (\mathrm{id})$$

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Logical rules for $\wedge$, $\vee$:

$$\frac{G\{\Sigma, A^\bullet, B^\bullet\}}{G\{\Sigma, A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \frac{G\{\Sigma, A^\circ\} \qquad G\{\Sigma, B^\circ\}}{G\{\Sigma, A \wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{G\{\Sigma, A^\bullet\} \qquad G\{\Sigma, B^\bullet\}}{G\{\Sigma, A \vee B^\bullet\}}\ \vee^\bullet \qquad \frac{G\{\Sigma, F^\circ\}}{G\{\Sigma, A \vee B^\circ\}}\ \vee^\circ, F \in \{A, B\}$$

The calculus $\mathbf{C}_{B^+}$: special rules

Logical rules for $\rightarrow$:

$$\frac{G\{\Sigma, \langle A^{\bullet} \mathbin{/\!\!/} B^{\circ}\rangle\}}{G\{\Sigma, A \rightarrow B^{\circ}\}} \rightarrow^{\circ}$$

$$\frac{G^{*}\{\Sigma^{*}, A \rightarrow B^{\bullet}, \langle \Phi_1^{*}, A^{\circ} \mathbin{/\!\!/} \Phi_2^{*}\rangle\} \qquad G\{\Sigma, A \rightarrow B^{\bullet}, \langle \Phi_1 \mathbin{/\!\!/} \Phi_2, B^{\bullet}\rangle\}}{G\{\Sigma, A \rightarrow B^{\bullet}, \langle \Phi_1 \mathbin{/\!\!/} \Phi_2\rangle\}} \rightarrow^{\bullet}$$

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A structural rule (to mirror monotonicity of $\leq$):

$$\frac{\Sigma, \langle A^\bullet, \Phi_1 \sslash A^\bullet, \Phi_2 \rangle}{\Sigma, \langle A^\bullet, \Phi_1 \sslash \Phi_2 \rangle} \text{ lift}$$

Example

$A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)^\circ$ is provable in $\mathbf{C}_{B^+}$.

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$$
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\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\langle A^\bullet, B^\bullet /\!\!/ A^\bullet, A^\circ \rangle$}
\RightLabel{lift}
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\RightLabel{$\to^\circ$}
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\end{prooftree}
$$

$\square$

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Compared with labeled calculi

Similar structural rules are necessarily included in a labeled calculus but do **NOT** need to be primitive here

Admissibility and invertibility

- A rule $\mathscr{R}$ is **admissible** if, when its premises are derivable at height n, the conclusion is derivable at height m. If $m \leq n$, the rule is **height-preserving admissible** (*hp-admissible*).
- A rule $\mathscr{R}$ is **height-preserving invertible** (*hp-invertible*) if, when the conclusion is derivable at height at most m, all premises are derivable at height at most m.

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Proposition

The rules R1 and R2 are hp-admissible in $\mathbf{C}_{\mathrm{B}^+}$.

Semantic interpretation

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For a **left-sided sequent** $\Sigma = A^\bullet_1, \ldots, A^\bullet_m, \langle \Phi_{11} /\!\!/ \Phi_{12} \rangle, \ldots, \langle \Phi_{j1} /\!\!/ \Phi_{j2} \rangle$, $x \Vdash \Sigma$ iff:

- for any $i \in \{1, \ldots, m\}$, $x \Vdash A^\bullet_i$; and
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- for any $i \in \{1, \ldots, m\}$, $x \Vdash A_i^\bullet$; and
- for any $k \in \{1, \ldots, j\}$, there exist y, z with $Rxyz$, $y \Vdash \Phi_{k1}$ and $z \Vdash \Phi_{k2}$

For a **full sequent**,

- $x \Vdash \Sigma, A^\circ$ iff $x \Vdash \Sigma$ implies $x \Vdash A$
- $x \Vdash \Sigma, \langle \Phi_1 /\!\!/ \Phi_2 \rangle$ (where Φ_1 is full) iff if $x \Vdash \Sigma$ then for any y, z with $Rxyz$, $z \Vdash \Phi_2$ implies $y \Vdash \Phi_1$
- $x \Vdash \Sigma, \langle \Phi_1 /\!\!/ \Phi_2 \rangle$ (where Φ_2 is full) iff if $x \Vdash \Sigma$ then for any y, z with $Rxyz$, $y \Vdash \Phi_1$ implies $z \Vdash \Phi_2$

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The satisfaction of a full sequent says if the left-sided part is satisfied, so is the unique output.

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According to the definition, for a model $\mathscr{M} = (\mathscr{O}, K, R, V)$ and $w \in \mathscr{O}$, if $\mathscr{M}, w \nVdash G\{\Sigma, A^\circ\}$, then there exists some $x \in K$ such that $x \nVdash \Sigma, A^\circ$.

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Notice that a block is interpreted differently depending on whether it contains the output formula and where that formula is precisely located:

- if a block **does not** contain the unique output formula, it is interpreted **existentially**
- if a block **does** contain the unique output formula, it is interpreted **universally**

Similar practice for interpreting nested sequents is also used in intuitionistic modal logic, e.g. [Straßburger, 2013, Gao and Olivetti, 2026]

Soundness of $\mathbf{C}_{\mathrm{B}^+}$

For a rule (r) in $\mathbf{C}_{\mathrm{B}^+}$, we say (r) is *valid* if whenever all of its premises are valid, the conclusion is also valid.

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Lemma

All the rules in $\mathbf{C}_{\mathrm{B}^+}$ are B^+-valid.

Theorem (Soundness of $\mathbf{C}_{\mathrm{B}^+}$)

If a nested sequent Σ is provable in $\mathbf{C}_{\mathrm{B}^+}$, then Σ is B^+-valid.

A syntactic proof of completeness of $\mathbf{C}_{\mathrm{B}^+}$ + cut

The cut rule can be applied to any component of the sequent:

$$\frac{G^*\{\Sigma^*, A^\circ\} \qquad G\{\Sigma, A^\bullet\}}{G\{\Sigma\}}\ \mathsf{cut}$$

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If a formula $A \in \mathrm{Frm}$ is valid, then it is provable in $\mathbf{C}_{\mathrm{B}^+}$ + cut.

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Theorem (Completeness of $\mathbf{C}_{\mathrm{B}^+}$ + cut)

If a formula $A \in \mathrm{Frm}$ is valid, then it is provable in $\mathbf{C}_{\mathrm{B}^+}$ + cut.

Proof.

For mp, we show that if $\vdash_{\mathbf{C}_{\mathrm{B}^+}} A \to B^\circ$ and $\vdash_{\mathbf{C}_{\mathrm{B}^+}} A^\circ$, then $\vdash_{\mathbf{C}_{\mathrm{B}^+}} B^\circ$.

$$\dfrac{A^\circ \qquad \dfrac{\dfrac{A \to B^\circ \qquad \dfrac{\dfrac{}{A \to B^\bullet, \langle A^\bullet, A^\circ /\!\!/ - \rangle}\,(\mathrm{id}) \qquad \dfrac{}{A \to B^\bullet, \langle A^\bullet /\!\!/ B^\bullet, B^\circ \rangle}\,(\mathrm{id})}{A \to B^\bullet, \langle A^\bullet /\!\!/ B^\circ \rangle}\,\to^\bullet}{\langle A^\bullet /\!\!/ B^\circ \rangle}\,\mathrm{cut}}{A^\bullet, B^\circ}\,\mathrm{R1}}{B^\circ}\,\mathrm{cut}$$

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Proof methodology

The proof proceeds by double induction on the **rank** of the cut, defined as the ordered pair:

$$\mathrm{Rank}(\mathrm{cut}) = (|A|, h_1 + h_2)$$

where $|A|$ represents the logical complexity of the cut formula, while h_1 and h_2 denote the heights of the derivations of the two premises.

Applications: verification lemma

Verification lemma

A formula $A \rightarrow B$ is satisfied (at a regular world) in a model $\mathscr{M}$ iff for all $a \in K$, $\mathscr{M}, a \Vdash A$ implies $\mathscr{M}, a \Vdash B$.

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Consider the following rules:

$$\dfrac{\langle A^{\bullet}, B^{\circ} \mathbin{/\!\!/} - \rangle}{A \longrightarrow B^{\circ}}\ \mathrm{V1} \qquad \dfrac{\langle - \mathbin{/\!\!/} A^{\bullet}, B^{\circ} \rangle}{A \longrightarrow B^{\circ}}\ \mathrm{V2}$$

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Corollary

The two rules V1, V2 *are valid and admissible in* $\mathbf{C}_{\mathrm{B}^+}$.

Applications: variable-sharing property

- A logic L is said to have the *variable-sharing property* (VSP for short) if $A \to B$ is a theorem in L, $\mathrm{Var}(A) \cap \mathrm{Var}(B) \neq \varnothing$.
- VSP is the intrinsic feature of relevance [Standefer, 2025]
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Let $\langle - /\!\!/ \Phi \rangle$ be a nested sequent with the output formula B. If it is provable, then there exist $\Sigma, \Psi \in^{+} \Phi$ such that

- $B^{\circ} \in \Psi$;
- *Σ is reachable to Ψ;*
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Theorem (VSP for $\mathbf{C}_{\mathrm{B}^{+}}$)

If $A \longrightarrow B^{\circ}$ *is provable in* $\mathbf{C}_{\mathrm{B}^{+}}$*, then* $\mathrm{Var}(A) \cap \mathrm{Var}(B) \neq \varnothing$.

Conclusion

In this work, we have

- proposed a *nested sequent calculus* directly internalizing Routley-Meyer ternary semantics using a novel notion of nested blocks
- interpreted sequents over B^+-frames and established soundness of the calculus
- established completeness by a syntactic proof of cut-elimination
- derived some metalogical properties as consequences of cut-free derivations

Further topics

Other obtained results (not included in the slides):

- extended the proof-theoretic framework to a modal logic $\mathsf{B}^{+}\mathsf{K}^{\Box}$
- extended the calculus to treat B with De Morgan negation: one more structural operator for $\sim$, a bi-nested flavor

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Ongoing work:

- treat stronger logics such as TW^+, RW^+: consider other structural rules
- relationship with other existing calculi
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Thank you for your attention!

Details are found at: `https://ghlogic123.github.io/nested_RL.pdf`

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