Basic zero-dimensional spaces: A unifying framework for continuity and openness

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Preimages: The case of locales

It is well known that, given a localic map $f\colon L \to K$ and a sublocale S of K, the standard set-theoretical preimage $f^{-1}[S]$ may not be a sublocale. Nevertheless, there exists *the largest sublocale contained in it*, which we denote by $f_{-1}[S]$. Moreover the map

$$\begin{aligned} f_{-1}\colon \mathsf{S}(K) &\longrightarrow \mathsf{S}(L) \\ S &\longrightarrow f_{-1}[S] \end{aligned}$$

is a coframe homomorphism.

Heyting semilattices

A *Heyting semilattice* H is a meet-semilattice in which every unary operator $\lambda_a = a \wedge (-)$ has a right adjoint $\gamma_a = a \rightarrow (-)$, that is,

$$a \wedge b \leq c \Leftrightarrow b \leq a \rightarrow c \quad \text{for every } a, b, c \in H.$$

In a Heyting semilattice H one has, for each $a \in H$, the sets

$$\mathfrak{o}a = \{a \rightarrow x \,|\, x \in H\} \quad \text{and} \quad \mathfrak{c}a = \uparrow a = \{x \in H \mid x \geq a\}.$$

They are called the opens and the closed, respectively.

Heyting semilattices

Definition

A map between Heyting semilattices is continuous if it has a left adjoint that preserves finite meets (including the top element).

In Heyting semilattices

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Definition

Given a Heyting semilattice H, a $N \subseteq H$ is called a nuclear range if it is the image of a nucleus.

Nuclear ranges generalize sublocales.
Nuclear ranges are not closed under binary intersections; in fact, they do not even need to have finite meets!
There is no non-complete analogue of the localic preimage!

Going back to locales

Proposition

Let L be a locale and $S \subseteq L$ a sublocale, then:

$$S = \bigcap\{\mathfrak{c}a \vee \mathfrak{o}b \mid S \subseteq \mathfrak{c}a \vee \mathfrak{o}b\}.$$

Substructures of H: The basic domains

Definition

Basic domains are the subsets of H of the form

$$\bigcap_{i \in I} (\mathfrak{c} a_i^1 \vee \mathfrak{o} b_i^1 \vee \cdots \vee \mathfrak{c} a_i^{n_i} \vee \mathfrak{o} b_i^{n_i}).$$

We denote the system of all basic domains of H, ordered by inclusion, by $\mathcal{B}d(H)$.

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Proposition

Given a Heyting semilattice H, $\mathcal{B}d(H)$ is a coframe.

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(BC2) *There exists a map* $f^{\mathfrak{c}}_{\leftarrow} : \mathfrak{c}H_2 \to \mathfrak{c}H_1$ *such that, for every* $b \in H_2$ *and* $a \in H_1$,

$$f[\mathfrak{c}a] \subseteq \mathfrak{c}b \Leftrightarrow \mathfrak{c}a \subseteq f^{\mathfrak{c}}_{\leftarrow}[\mathfrak{c}b].$$

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(BC3) *For every $b \in H_2$, $\neg f^{\mathfrak{c}}_{\leftarrow}[\mathfrak{c}b] \subseteq f^{-1}[\mathfrak{o}b]$.*

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(BC3) *For every $b \in H_2$, $\neg f^{\mathfrak{c}}_{\leftarrow}[\mathfrak{c}b] \subseteq f^{-1}[\mathfrak{o}b]$.*

(BC4) *For every $a_1, \ldots, a_n,\ b_1, \ldots, b_n \in H_2$,*

$$\bigvee_{i=1}^{n} (f^{\mathfrak{c}}_{\leftarrow}[\mathfrak{c}a_i] \vee \neg f^{\mathfrak{c}}_{\leftarrow}[\mathfrak{c}b_i]) \subseteq f^{-1}[\bigvee_{i=1}^{n} (\mathfrak{c}a_i \vee \mathfrak{o}b_i)].$$

Basic continuous maps

We say that a map is *basic continuous* if it satisfies conditions (BC1) to (BC4).

Basic zero-dimensional spaces

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A *basic zero-dimensional space* (briefly, *b0-space*) is a triple

$$\mathcal{S} = (X, \mathcal{C}, \mathcal{B}d)$$

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- X is a set,
- $\mathcal{B}d$ is a closure system on X, distributive as a lattice, and
- $\mathcal{C} \subseteq \mathcal{B}d$ is such that every $C \in \mathcal{C}$ has a complement $\neg C$ in $\mathcal{B}d$, and every $M \in \mathcal{B}d$ is of the form

$$M = \bigcap_{i \in J} (C_i^1 \vee \neg D_i^1 \vee \cdots \vee C_i^{n_i} \vee \neg D_i^{n_i}),$$

for some $C_i^1, D_i^1, \ldots, C_i^{n_i}, D_i^{n_i} \in \mathcal{C}$.

Basic zero-dimensional spaces

The bottom element of $\mathcal{B}d$ is denoted by O, and we define $\mathcal{O} = \{\neg C \mid C \in \mathcal{C}\}$.
We denote the underlying set of $\mathcal{S}$ by $|\mathcal{S}|$.

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Proposition

If $\mathcal{S} = (X, \mathcal{C}, \mathcal{B}d)$ is a basic zero-dimensional space, then $\mathcal{B}d$ is a coframe.

Basic continuous maps

Definition

Let $\mathcal{S}$ and $\mathcal{T}$ be b0-spaces. A *basic continuous map* $f\colon \mathcal{S} \to \mathcal{T}$ is a map $f\colon |\mathcal{S}| \to |\mathcal{T}|$ that satisfies the following conditions:

(BC1) $f^{-1}[\mathrm{O}] = \mathrm{O}$.

(BC2) There exists a map $f^{\mathfrak{c}}_{\leftarrow}\colon \mathcal{C}(\mathcal{T}) \to \mathcal{C}(\mathcal{S})$, such that, for every $B \in \mathcal{C}(\mathcal{S})$ and $C \in \mathcal{C}(\mathcal{T})$,

$$f[B] \subseteq C \Leftrightarrow B \subseteq f^{\mathfrak{c}}_{\leftarrow}[C].$$

(BC3) For every $C \in \mathcal{C}(\mathcal{T})$, $\neg f^{\mathfrak{c}}_{\leftarrow}[C] \subseteq f^{-1}[\neg C]$.

(BC4) For every $C_i,\ D_i \in \mathcal{C}(\mathcal{T})$,

$$\bigvee_{i=1}^{n} (f^{\mathfrak{c}}_{\leftarrow}[C_i] \vee \neg f^{\mathfrak{c}}_{\leftarrow}[D_i]) \subseteq f^{-1}[\bigvee_{i=1}^{n} (C_i \vee \neg D_i)].$$

We denote the category of basic zero-dimensional spaces and basic continuous maps by

$$\mathbf{B0ds}.$$

Examples

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- The category of closure spaces.

Examples

All the following categories are embeddable in **B0ds**:

- The category of Heyting semilattices and basic continuous maps (in particular, the category **Loc**).
- The category of topological spaces and continuous maps, **Top**.
- The category of closure spaces.
- The category of measurable space and measurable functions.

Properties of basic continuous maps

Theorem (Extension)

Let $\mathcal{S}$ and $\mathcal{T}$ be two b0-spaces. A map $f : |\mathcal{S}| \to |\mathcal{T}|$ is basic continuous if and only if it satisfies the following properties:

Properties of basic continuous maps

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(BC2') *There exists a map $f_{\leftarrow} : \mathcal{B}d(\mathcal{T}) \to \mathcal{B}d(\mathcal{S})$ such that, for every $M \in \mathcal{B}d(\mathcal{S}), K \in \mathcal{B}d(\mathcal{T})$,*

$$f[M] \subseteq K \Leftrightarrow M \subseteq f_{\leftarrow}[K].$$

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(BC4') *For every $C \in \mathcal{C}(\mathcal{T})$, $f_{\leftarrow}[\neg C] = \neg f_{\leftarrow}[C]$.*

Properties of basic continuous maps

We call

$$f_{\leftarrow}$$

the *basic preimage* of f, and say that

$$(f, f_{\leftarrow})$$

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Theorem

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Corollary

Let $(f, f_{\leftarrow})$ be a basic continuous pair. Then $f_{\leftarrow}$ preserves complements.

Returning to Heyting semilattices

Corollary

Let $f : H_1 \to H_2$ be a continuous map between Heyting semilattices. For each $K \in \mathcal{B}d(H_2)$, there exists the largest basic domain contained in $f^{-1}[K]$, denoted $f_{\leftarrow}[K]$. Moreover, the map

$$\begin{aligned} f_{\leftarrow} : \mathcal{B}d(H_2) &\longrightarrow \mathcal{B}d(H_1) \\ K &\longmapsto f_{\leftarrow}[K] \end{aligned}$$

is a coframe homomorphism satisfying $f_{\leftarrow}[\mathfrak{o}b] = \mathfrak{o}(f^*(b)) = \neg f^{-1}[\mathfrak{c}b]$.

Basic continuous maps

The image of a basic domain by a basic continuous map does not need to be a basic domain.
However, since $\mathcal{B}d$ is a closure system, it is associated with the closure operator given by

$$\begin{aligned} \langle - \rangle : \mathcal{P}(X) &\longrightarrow \mathcal{B}d \\ M &\longmapsto \langle M \rangle = \bigcap \{S \in \mathcal{B}d \mid M \subseteq S\}. \end{aligned} \tag{Cl Bd}$$

Using this, we may associate with each basic continuous map $f \colon \mathcal{S} \to \mathcal{T}$ the map

$$\begin{aligned} f_{\to} \colon \mathcal{B}d(\mathcal{S}) &\longrightarrow \mathcal{B}d(\mathcal{T}) \\ M &\longmapsto \langle f[M] \rangle \end{aligned} \tag{BI}$$

which will be called the *basic image* of f.

Open maps

Definition

A basic continuous map $f : \mathcal{S} \to \mathcal{T}$ is said to be a *basic open map* if, for every $U \in \mathcal{O}(\mathcal{S})$, $f_{\rightarrow}[U]$ is still an open.

Open maps in b0-spaces

Given a b0-space $\mathcal{S} = (X, \mathcal{C}, \mathcal{B}d)$, let

$$\mathcal{O}^{\mathfrak{f}}(\mathcal{S}) = \bigvee^{\mathfrak{f}} \bigwedge^{\mathfrak{f}} \mathcal{O}(\mathcal{S}).$$

We will refer to the elements of $\mathcal{O}^{\mathfrak{f}}(\mathcal{S})$ as *finitely generated open*.

Open maps in b0-spaces

Given a b0-space $\mathcal{S} = (X, \mathcal{C}, \mathcal{B}d)$, let

$$\mathcal{O}^{\mathfrak{f}}(\mathcal{S}) = \mathsf{V}^{\mathfrak{f}}\Lambda^{\mathfrak{f}}\mathcal{O}(\mathcal{S}).$$

We will refer to the elements of $\mathcal{O}^{\mathfrak{f}}(\mathcal{S})$ as *finitely generated open*.

Further, given a basic continuous pair $(f, f_{\leftarrow})$ between b0-spaces $\mathcal{S}$ and $\mathcal{T}$, we will refer to the map

$$\begin{aligned} f_{\leftarrow}^{\mathfrak{fo}} : \mathcal{O}^{\mathfrak{f}}(\mathcal{T}) &\longrightarrow \mathcal{O}^{\mathfrak{f}}(\mathcal{S}) \\ V &\longmapsto f_{\leftarrow}[V] \end{aligned}$$

as the *finite open restriction* of $f_{\leftarrow}$.

Open maps in b0-spaces

Theorem

Let $\mathcal{S}$ be a b0-space such that $\Lambda^{\mathfrak{f}}\mathcal{O}(\mathcal{S}) = \mathcal{O}(\mathcal{S})$ and let $\mathcal{T}$ be an arbitrary b0-space. Let also $(f, f_{\leftarrow})$ be a basic continuous pair between $\mathcal{S}$ and $\mathcal{T}$. Then f is a basic open map if and only if $f_{\leftarrow}^{\mathfrak{fo}}$ has a left adjoint

$$h \colon \mathcal{O}^{\mathfrak{f}}(\mathcal{S}) \to \mathcal{O}^{\mathfrak{f}}(\mathcal{T})$$

such that the following conditions hold:

1. *For every $U \in \mathcal{O}(\mathcal{S})$, $h[U] \in \mathcal{O}(\mathcal{T})$.*
2. *For every $U \in \mathcal{O}(\mathcal{S})$ and $V_1, \ldots, V_n \in \mathcal{O}(\mathcal{T})$,*

$$h[U \cap f_{\leftarrow}[V_1 \cap \cdots \cap V_n]] = h[U] \cap V_1 \cap \cdots \cap V_n.$$

Open maps in b0-spaces

Proposition

Let $\mathcal{S}$ and $\mathcal{T}$ be b0-spaces such that $\Lambda^{\mathfrak{f}}\mathcal{O}(\mathcal{S}) = \mathcal{O}(\mathcal{S})$ and $\Lambda^{\mathfrak{f}}\mathcal{O}(\mathcal{T}) = \mathcal{O}(\mathcal{T})$. Let also $(f, f_{\leftarrow})$ be a basic continuous pair between $\mathcal{S}$ and $\mathcal{T}$ such that, for every $U \in \mathcal{O}(\mathcal{S})$,

$$f_{\rightarrow}[U] = \bigcap\{\neg V \vee W \mid f_{\rightarrow}[U] \subseteq \neg V \vee W \text{ and } V, W \in \mathcal{O}(\mathcal{T})\}.$$

Then f is a basic open map if and only if the map

$$f_{\leftarrow}^{\mathfrak{o}} = (V \mapsto f_{\leftarrow}[V]) \colon \mathcal{O}(\mathcal{T}) \to \mathcal{O}(\mathcal{S})$$

has a left adjoint $h \colon \mathcal{O}(\mathcal{S}) \to \mathcal{O}(\mathcal{T})$ such that

$$h[U \cap f_{\leftarrow}[V]] = h[U] \cap V \quad \text{for every } U \in \mathcal{O}(\mathcal{S}) \text{ and } V \in \mathcal{O}(\mathcal{T}).$$

Open maps in Heyting semilattices

Theorem

Let $f\colon H_1 \to H_2$ be a continuous map between Heyting semilattices. Then the following are equivalent:

(i) f *is open.*

(ii) *The map* $f_{\leftarrow}^{\mathfrak{o}} = (\mathfrak{o}b \mapsto f_{\leftarrow}[\mathfrak{o}b])\colon \mathfrak{o}H_2 \to \mathfrak{o}H_1$ *has a left adjoint* $h\colon \mathfrak{o}H_1 \to \mathfrak{o}H_2$ *such that*

$$h[\mathfrak{o}a \cap f_{\leftarrow}[\mathfrak{o}b]] = h[\mathfrak{o}a] \cap \mathfrak{o}b \quad \text{for every } a \in H_1 \text{ and } b \in H_2.$$

Open maps in Heyting semilattices

$$\mathfrak{o}H_1 \xrightarrow{\quad h \quad} \mathfrak{o}H_2 \xrightarrow{\quad f^{\mathfrak{o}}_{\leftarrow} \quad} \mathfrak{o}H_1$$

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Because $f_{\leftarrow}[\mathfrak{o}b] = \mathfrak{o}(f^*(b))$.

Open maps in Heyting semilattices

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\end{tikzcd}$$

Now, we use the fact that $\mathfrak{o}\colon H \to \mathfrak{o}H$ is an isomorphism and the commutativity of the previous diagram to deduce that $f^{\mathfrak{o}}_{\leftarrow}$ has a left adjoint $h\colon \mathfrak{o}H_1 \to \mathfrak{o}H_2$ if and only if $f^*\colon H_1 \to H_2$ has a left adjoint $f_!\colon H_1 \to H_2$.

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$$h[\mathfrak{o}a \cap f^{\mathfrak{o}}_{\leftarrow}[\mathfrak{o}b]] = h[\mathfrak{o}a] \cap \mathfrak{o}b \Leftrightarrow f_!(a \wedge f^*(b)) = f_!(a) \wedge b.$$

Open maps in Heyting semilattices

Theorem

Let $f: H_1 \to H_2$ be a continuous map between Heyting semilattices. Then the following are equivalent:

(i) *f is open.*

(ii) *The map $f^{\mathfrak{o}}_{\leftarrow} = (\mathfrak{o}b \mapsto f_{\leftarrow}[\mathfrak{o}b])\colon \mathfrak{o}H_2 \to \mathfrak{o}H_1$ has a left adjoint $h\colon \mathfrak{o}H_1 \to \mathfrak{o}H_2$ such that*

$$h[\mathfrak{o}a \cap f_{\leftarrow}[\mathfrak{o}b]] = h[\mathfrak{o}a] \cap \mathfrak{o}b \quad \text{for every } a \in H_1 \text{ and } b \in H_2.$$

(iii) *f^* has a left adjoint $f_!: H_1 \to H_2$ such that $f_!(a \wedge f^*(b)) = f_!(a) \wedge b$ for every $a \in H_1$ and $b \in H_2$.*

Open maps in Heyting semilattices

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Let $f\colon H_1 \to H_2$ be a continuous map between Heyting semilattices. Then the following are equivalent:

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(iii) f^* *has a left adjoint* $f_!\colon H_1 \to H_2$ *such that* $f_!(a \wedge f^*(b)) = f_!(a) \wedge b$ *for every* $a \in H_1$ *and* $b \in H_2$.

(iv) f^* *is a right adjoint such that* $f^*(b \to c) = f^*(b) \to f^*(c)$ *for every* $b, c \in H_2$.

A duality

Let $\mathcal{S} = (X, \mathcal{C}(\mathcal{S}), \mathcal{B}d(\mathcal{S}))$ be a basic zero-dimensional space. From it we can construct a new b0-space

$$\widetilde{\mathcal{S}} = (X, \mathcal{O}(\mathcal{S}), \mathcal{B}d(\mathcal{S})),$$

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which we call the *dual* of $\mathcal{S}$.

Given a morphism $f\colon \mathcal{S} \to \mathcal{T}$ between b0-spaces, we denote by $|f|$ the map between their underlying sets. Clearly, $f\colon \mathcal{S} \to \mathcal{T}$ is totally determined by $|f|$. So we can define $\widetilde{f}\colon \widetilde{\mathcal{S}} \to \widetilde{\mathcal{T}}$ such that $|\widetilde{f}| = |f|$.

Closed maps

Definition

A basic continuous map $f : \mathcal{S} \to \mathcal{T}$ between b0-spaces is *basic closed* if

$$f_{\to}[B] \in \mathcal{C}(\mathcal{T}) \text{ for every } B \in \mathcal{C}(\mathcal{S}).$$

Closed maps

Now, given a b0-space $\mathcal{S} = (X, \mathcal{C}, \mathcal{B}d)$, set

$$\mathcal{C}^{\mathfrak{f}}(\mathcal{S}) = \mathsf{V}^{\mathfrak{f}}\Lambda^{\mathfrak{f}}\mathcal{C}(\mathcal{S}).$$

Its elements are called *finitely generated closed domains*. Further, given a basic continuous pair $(f, f_{\leftarrow})$ between b0-spaces $\mathcal{S}$ and $\mathcal{T}$, we will refer to the map

$$\begin{aligned} f_{\leftarrow}^{\mathfrak{fc}} \colon \mathcal{C}^{\mathfrak{f}}(\mathcal{T}) &\longrightarrow \mathcal{C}^{\mathfrak{f}}(\mathcal{S}) \\ C &\longmapsto f_{\leftarrow}[C] \end{aligned}$$

as the *finite closed restriction of* $f_{\leftarrow}$.

Basic closed maps

Theorem

Let $\mathcal{S}$ be a b0-space such that $\Lambda^{\mathfrak{f}}\mathcal{C}(\mathcal{S}) = \mathcal{C}(\mathcal{S})$ and let $\mathcal{T}$ be an arbitrary b0-space. Let $(f, f_{\leftarrow})$ be a basic continuous pair between $\mathcal{S}$ and $\mathcal{T}$. Then f is a basic closed map if and only if $f_{\leftarrow}^{\mathfrak{fc}}$ has a left adjoint $h\colon \mathcal{C}^{\mathfrak{f}}(\mathcal{S}) \to \mathcal{C}^{\mathfrak{f}}(\mathcal{T})$ with the following properties:

1. *For every $C \in \mathcal{C}(\mathcal{S})$, $h[C] \in \mathcal{C}(\mathcal{T})$.*
2. *For every $C \in \mathcal{C}(\mathcal{S})$ and every $D_1, \ldots, D_n \in \mathcal{C}(\mathcal{T})$*

$$h[C \cap f_{\leftarrow}^{\mathfrak{fc}}[D_1 \cap \cdots \cap D_n]] = h[C] \cap D_1 \cap \cdots \cap D_n.$$

Closed maps Heyting algebras

Theorem

The following are equivalent for a continuous map $f: H_1 \to H_2$ between Heyting algebras:

(i) *f is a closed map.*

(ii) *The basic $\mathfrak{c}$-preimage $f_{\leftarrow}^{\mathfrak{c}}$ has a left adjoint $h\colon \mathfrak{c}H_1 \to \mathfrak{c}H_2$ such that*

$$h[\mathfrak{c}a \cap f_{\leftarrow}[\mathfrak{c}b]] = h[\mathfrak{c}a] \cap \mathfrak{c}b \quad \text{for every } a \in H_1 \text{ and } b \in H_2.$$

Closed maps in Heyting algebras

$$\begin{tikzcd}
\mathfrak{c}H_1 \arrow[r, "h"] & \mathfrak{c}H_2 \arrow[r, "f^{\mathfrak{c}}_{\leftarrow}"] & \mathfrak{c}H_1 \\
H_1 \arrow[u, "\mathfrak{c}"] \arrow[r, "f_!"] & H_2 \arrow[u, "\mathfrak{c}"] \arrow[r, "f^*"] & H_1 . \arrow[u, "\mathfrak{c}"]
\end{tikzcd}$$

Now we use the fact that $\mathfrak{c}\colon H \to \mathfrak{c}H$ is an anti-isomorphism to conclude that $f^{\mathfrak{c}}_{\leftarrow}$ has a left adjoint $h\colon \mathfrak{c}H_1 \to \mathfrak{c}H_2$ if and only if $f^*\colon H_2 \to H_1$ has a right adjoint $f_!\colon H_1 \to H_2$.

Closed maps in Heyting algebras

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\mathfrak{c}H_1 \arrow[r, "h"] & \mathfrak{c}H_2 \arrow[r, "f^{\mathfrak{c}}_{\leftarrow}"] & \mathfrak{c}H_1 \\
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\end{tikzcd}$$

Now we use the fact that $\mathfrak{c}\colon H \to \mathfrak{c}H$ is an anti-isomorphism to conclude that $f^{\mathfrak{c}}_{\leftarrow}$ has a left adjoint $h\colon \mathfrak{c}H_1 \to \mathfrak{c}H_2$ if and only if $f^*\colon H_2 \to H_1$ has a right adjoint $f_!\colon H_1 \to H_2$. Moreover

$$h[\mathfrak{c}a \cap f^{\mathfrak{c}}_{\leftarrow}[\mathfrak{c}b]] = h[\mathfrak{c}a] \cap \mathfrak{c}b \Leftrightarrow f(a \vee f^*(b)) = f(a) \vee b.$$

Closed maps Heyting algebras

Theorem

The following are equivalent for a continuous map $f\colon H_1 \to H_2$ between Heyting lattices:

1. (i) *f is a closed map.*
2. (ii) *The basic $\mathfrak{c}$-preimage $f^{\mathfrak{c}}_{\leftarrow}$ has a left adjoint $h\colon \mathfrak{c}H_1 \to \mathfrak{c}H_2$ such that*

$$h[\mathfrak{c}a \cap f_{\leftarrow}[\mathfrak{c}b]] = h[\mathfrak{c}a] \cap \mathfrak{c}b \quad \text{for every } a \in H_1 \text{ and } b \in H_2.$$

3. (iii) *For every $a \in H_1$ and $b \in H_2$, $f(a \vee f^*(b)) = f(a) \vee b$.*