# A Directed Refinement of Constructive Ordinals in Algebraic Set Theory

Benno van den Berg, Josje van der Laan

Topology, Algebra and Categories in Logic (TACL)
28 July 2026, Kraków

Institute for Logic, Language and Computation

# Table of Contents

# Constructive Ordinals

Ordinals have a rich structure and important applications:

- induction and recursion
- transfinite constructions
- logical consistency
- program termination

# Constructive Ordinals

Ordinals have a rich structure and important applications:

- induction and recursion
- transfinite constructions
- logical consistency
- program termination

**Question**

How can we define ordinals in a constructive setting?

# Approaches to constructive ordinals

Ordinals have be studied in several constructive settings.

- Constructive Zermelo Fraenkel (CZF)
- Homotopy Type Theory (HoTT)
- Algebraic Set Theory (AST)

# Constructive Zermelo Fraenkel

**CZF:** constructive and predicative variant of ZF Set Theory.

# Constructive Zermelo Fraenkel

**CZF:** constructive and predicative variant of ZF Set Theory.

- **Constructive:** underlying logic is intuitionistic, constructive formulation of axioms.
- **Predicative:** avoids impredicative principles such as the full Power Set axiom.

# Ordinals in CZF

**Definition**

A set $x$ is **transitive** if

$$z \in y \text{ and } y \in x \ \Rightarrow\ z \in x,$$

i.e., if $y \in x$ then $y \subseteq x$.

# Ordinals in CZF

**Definition**

A set $x$ is **transitive** if

$$z \in y \text{ and } y \in x \;\Rightarrow\; z \in x,$$

i.e., if $y \in x$ then $y \subseteq x$.

**Definition**

An **ordinal** is a transitive set of transitive sets, so a *hereditarily transitive* set (Powell 1975).

# Ordinals in CZF

**Definition**

A set $x$ is **transitive** if

$$z \in y \text{ and } y \in x \;\Rightarrow\; z \in x,$$

i.e., if $y \in x$ then $y \subseteq x$.

**Definition**

An **ordinal** is a transitive set of transitive sets, so a *hereditarily transitive* set (Powell 1975).

{Powell, **Grayson**, Von Neumann} Ordinals.

# Structure I

**Definition**

We define $<$ and $\leq$: for any two ordinals $\alpha, \beta$:

$$\alpha < \beta \iff \alpha \in \beta,$$

$$\alpha \leq \beta \iff \alpha \subseteq \beta.$$

# Structure I

## Definition

We define $<$ and $\leq$: for any two ordinals $\alpha, \beta$:

$$\alpha < \beta \iff \alpha \in \beta,$$

$$\alpha \leq \beta \iff \alpha \subseteq \beta.$$

## Proposition (Grayson, 1977)

*Any of the following would imply LEM:*

- $\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha,$
- $\alpha \leq \beta \vee \beta \leq \alpha,$
- $\alpha \leq \beta \implies \alpha < \beta \vee \alpha = \beta.$

# Structure II

**Definition**

For an ordinal $\alpha$ and a set of ordinals $A$, we define the **successor**:

$$\bar{s}(\alpha) = \alpha \cup \{\alpha\},$$

and the **supremum**:

$$\bigvee A = \bigcup A.$$

# Structure II

**Definition**

For an ordinal $\alpha$ and a set of ordinals $A$, we define the **successor**:

$$\bar{s}(\alpha) = \alpha \cup \{\alpha\},$$

and the **supremum**:

$$\bigvee A = \bigcup A.$$

**Proposition (Grayson, 1977)**

*The assertion that every ordinal is either zero, a successor, or a limit ordinal implies LEM.*

# Structure III

**Definitions**

A relation $<$ on $A$ is **well-founded** if:

$$\forall X \subseteq A \left[(\forall y\, (\forall x < y : x \in X) \Rightarrow y \in X) \Rightarrow X = A\right].$$

A **well-order** is a well-founded relation that is:

- transitive,
- extensional.

**Theorem (Grayson, 1977)**

*$(A, <)$ is a well-order $\iff$ for some ordinal $\alpha$: $(A, <) \cong (\alpha, \in)$.*

# Ordinals in HoTT

HoTT: Homotopy Type Theory.
$\rightarrow$ constructive and predicative foundation.

**Definition**

An **ordinal** is a type with an order relation $<$ that is *extensional*, *transitive*, and *well-founded* (The Univalent Foundations Program 2013).

# Ordinals in HoTT

HoTT: Homotopy Type Theory.
→ constructive and predicative foundation.

**Definition**

An **ordinal** is a type with an order relation $<$ that is *extensional*, *transitive*, and *well-founded* (The Univalent Foundations Program 2013).

**Result (De Jong et al., 2023)**

CZF ordinals coincide with HoTT ordinals as types.

# Ordinals in a Categorical Framework

The notion of ordinal in CZF and HoTT can be translated into the framework of *Algebraic Set Theory* (AST).

# Ordinals in a Categorical Framework

The notion of ordinal in CZF and HoTT can be translated into the framework of *Algebraic Set Theory* (AST).

**Algebraic Set Theory**: categorical framework to study set-theoretic structures (Joyal and Moerdijk 1995).

# Ordinals in a Categorical Framework

The notion of ordinal in CZF and HoTT can be translated into the framework of *Algebraic Set Theory* (AST).

**Algebraic Set Theory**: categorical framework to study set-theoretic structures (Joyal and Moerdijk 1995).

**Idea**

Describe ordinals as an **initial algebraic structure**.

## ZF-algebras

**Definition**

A **ZF-algebra** is a structure $(A, \leq, s)$ where:

- $(A, \leq)$ is a partially ordered class with all small suprema,

# ZF-algebras

## Definition

A **ZF-algebra** is a structure $(A, \leq, s)$ where:

- $(A, \leq)$ is a partially ordered class with all small suprema,
- $s : A \to A$ is a map called the *successor*.

# ZF-algebras

## Definition

A **ZF-algebra** is a structure $(A, \leq, s)$ where:

- $(A, \leq)$ is a partially ordered class with all small suprema,
- $s : A \to A$ is a map called the *successor*.

A **morphism** of ZF-algebras is a map $f : A \to B$ that preserves all small suprema and the successor:

# ZF-algebras

## Definition

A **ZF-algebra** is a structure $(A, \leq, s)$ where:

- $(A, \leq)$ is a partially ordered class with all small suprema,
- $s : A \to A$ is a map called the *successor*.

A **morphism** of ZF-algebras is a map $f : A \to B$ that preserves all small suprema and the successor:

$$f(\bigvee^{A} X) = \bigvee^{B}_{x \in X} f(x), \quad f(s_A(x)) = s_B(f(x)).$$

# ZF-algebras

**Definition**

A **ZF-algebra** is a structure $(A, \leq, s)$ where:

- $(A, \leq)$ is a partially ordered class with all small suprema,
- $s : A \to A$ is a map called the *successor*.

A **morphism** of ZF-algebras is a map $f : A \to B$ that preserves all small suprema and the successor:

$$f(\bigvee^{A} X) = \bigvee^{B}_{x \in X} f(x), \quad f(s_A(x)) = s_B(f(x)).$$

The category **ZFAlg** has ZF-algebras as objects and morphisms of ZF-algebras as arrows.

# Example: (Initial) ZF-algebra

## Example

The class of all sets $V$ forms a ZF-algebra, with:

- Order: inclusion $\subseteq$,
- Suprema: unions $\bigcup$,
- Successor: $s(x) = \{x\}$.

# Example: (Initial) ZF-algebra

**Example**

The class of all sets $V$ forms a ZF-algebra, with:

- Order: inclusion $\subseteq$,
- Suprema: unions $\bigcup$,
- Successor: $s(x) = \{x\}$.

This structure is the *free ZF-algebra on the empty set*, i.e., the initial ZF-algebra.

# Example: (Initial) ZF-algebra

## Example

The class of all sets $V$ forms a ZF-algebra, with:

- Order: inclusion $\subseteq$,
- Suprema: unions $\bigcup$,
- Successor: $s(x) = \{x\}$.

This structure is the *free ZF-algebra on the empty set*, i.e., the initial ZF-algebra.

We can define a membership relation $\in$ by:

$$a \in b \iff s(a) = \{a\} \subseteq b.$$

# Example: (Initial) ZF-algebra

## Example

The class of all sets $V$ forms a ZF-algebra, with:

- Order: inclusion $\subseteq$,
- Suprema: unions $\bigcup$,
- Successor: $s(x) = \{x\}$.

This structure is the *free ZF-algebra on the empty set*, i.e., the initial ZF-algebra.

We can define a membership relation $\in$ by:

$$a \in b \iff s(a) = \{a\} \subseteq b.$$

$(V, \in)$ provides a model for ZF set theory.

# Example: Initial with Inflationary Successor

**Definition**

A successor $s : A \to A$ is **inflationary** if

$$x \leq s(x) \quad \text{for all } x \text{ in } A.$$

# Example: Initial with Inflationary Successor

**Definition**

A successor $s : A \to A$ is **inflationary** if

$$x \leq s(x) \quad \text{for all } x \text{ in } A.$$

Let $\mathbf{ZFAlg}_{\text{infl}}$ be the full subcategory of **ZFAlg** whose objects $(A, \leq, s)$ have an inflationary successor.

# Example: Initial with Inflationary Successor

## Definition

A successor $s : A \to A$ is **inflationary** if

$$x \leq s(x) \quad \text{for all } x \text{ in } A.$$

Let $\mathbf{ZFAlg}_{\mathrm{infl}}$ be the full subcategory of **ZFAlg** whose objects $(A, \leq, s)$ have an inflationary successor.

## Result (Joyal and Moerdijk, 1995)

The **initial ZF-algebra with an inflationary successor** is:

$$(T^2, \subseteq, \bar{s})$$

where $T^2$ is the class of hereditarily transitive sets, $\bar{s}(x) = x \cup \{x\}$ and suprema are given by unions.

# Strict Order

**Definition**

For a ZF-algebra $(A, \leq, s)$, define a strict order $<$ by:

$$x < y \iff s(x) \leq y.$$

# Strict Order

**Definition**

For a ZF-algebra $(A, \leq, s)$, define a strict order $<$ by:

$$x < y \iff s(x) \leq y.$$

**Example**

1. For $(V, \subseteq, \{\cdot\})$, the strict order $<$ coincides with $\in$.

# Strict Order

**Definition**

For a ZF-algebra $(A, \leq, s)$, define a strict order $<$ by:

$$x < y \iff s(x) \leq y.$$

**Example**

1. For $(V, \subseteq, \{\cdot\})$, the strict order $<$ coincides with $\in$.
2. For Grayson ordinals $(T^2, \subseteq, \bar{s})$, $<$ also coincides with $\in$:

$$\bar{s}(\alpha) = \alpha \cup \{\alpha\} \subseteq \beta \iff \alpha \in \beta.$$

# Tarski Ordinals I

**Definition**

A subset $D \subseteq A$ is **weakly directed** if for any $\alpha, \beta \in D$, there exists $\gamma \in D$ such that

$$\alpha, \beta \leq \gamma.$$

**Definition**

A successor $s : A \to A$ is **monotone** if, for all $\alpha, \beta \in A$,

$$\alpha \leq \beta \implies s(\alpha) \leq s(\beta).$$

# Tarski Ordinals I

**Definition**

A subset $D \subseteq A$ is **weakly directed** if for any $\alpha, \beta \in D$, there exists $\gamma \in D$ such that

$$\alpha, \beta \leq \gamma.$$

**Definition**

A successor $s : A \to A$ is **monotone** if, for all $\alpha, \beta \in A$,

$$\alpha \leq \beta \implies s(\alpha) \leq s(\beta).$$

**Tarski ordinals:** free partially ordered structure with all weakly directed suprema and a monotone successor.

- Forms ZF-algebra.
- Used to prove Tarski's Fixed Point Theorem.

# Tarski Ordinals II

**Characterisation Tarski Ordinals**

Initial ZF-algebra whose successor preserves binary suprema.

# Tarski Ordinals II

**Characterisation Tarski Ordinals**

Initial ZF-algebra whose successor preserves binary suprema.

**Property**

These ordinals are (weakly) **directed**:

$$\forall \alpha, \beta < \gamma \;\Rightarrow\; \exists \delta < \gamma \text{ with } \alpha, \beta \leq \delta,$$

where $<$ is the order induced by $s$ and $\leq$.

$$\begin{tikzcd} \alpha \arrow[dr, "\leq"] & \\ & \delta \\ \beta \arrow[ur, "\leq"'] & \end{tikzcd}$$

# Directedness

- **Directedness**: any two ordinals have a common refinement.
- Limit stages in transfinite constructions are **filtered colimits**.

# Why an inflationary successor?

Tarski ordinals use **monotone** successor.

- Because of monotonicity,

$$2 = \{0, 1\} \text{ is an ordinal} \iff \text{LEM}.$$

  In impredicative setting, take $2 = \mathcal{P}1$. Predicatively, not so desirable...

# Why an inflationary successor?

Tarski ordinals use **monotone** successor.

- Because of monotonicity,

$$2 = \{0, 1\} \text{ is an ordinal} \iff \text{LEM}.$$

  In impredicative setting, take $2 = \mathcal{P}1$. Predicatively, not so desirable...
- Successor in CZF and HoTT **only inflationary**.

# Why an inflationary successor?

Tarski ordinals use **monotone** successor.

- Because of monotonicity,

$$2 = \{0, 1\} \text{ is an ordinal} \iff \text{LEM}.$$

  In impredicative setting, take $2 = \mathcal{P}1$. Predicatively, not so desirable...

- Successor in CZF and HoTT **only inflationary**.

Can we recover directedness using only an inflationary successor?

# Limitation of Inflationary Successor

It follows from the result in (Bauer and Lumsdaine 2013) that for the free partially ordered structure with an **inflationary successor** and only **weakly directed suprema**, (constructively) one **cannot** guarantee existence of all suprema.

# Limitation of Inflationary Successor

It follows from the result in (Bauer and Lumsdaine 2013) that for the free partially ordered structure with an **inflationary successor** and only **weakly directed suprema**, (constructively) one **cannot** guarantee existence of all suprema.

- So we cannot directly mimic the construction of Tarski ordinals.
- Need a different approach to obtain directed ordinals with inflationary successor.

# Downset of ordinals

## Definition

For $\alpha$, define the (strict) downclass:

$$\Downarrow \alpha = \{ \beta \mid \beta < \alpha \}.$$

# Downset of ordinals

**Definition**

For $\alpha$, define the (strict) downclass:

$$\Downarrow \alpha = \{ \beta \mid \beta < \alpha \}.$$

**Proposition**

Let $N$ be the initial ZF-algebra with inflationary successor. For any ordinal $\alpha$ in $N$ we have that $\Downarrow \alpha$ is a set.

# Directed Refinement

**Definition**

We consider full subcategory

$$\mathcal{D} := \mathbf{ZFAlg}_{\text{infl,dir}}$$

of ZF-algebras with inflationary successors in which all downclasses are closed under binary suprema:

$$\forall x, y \in \Downarrow a, \quad x \vee y \in \Downarrow a.$$

# Directed Refinement

**Definition**

We consider full subcategory

$$\mathcal{D} := \mathbf{ZFAlg}_{\mathrm{infl,dir}}$$

of ZF-algebras with inflationary successors in which all downclasses are closed under binary suprema:

$$\forall x, y \in \Downarrow a, \quad x \vee y \in \Downarrow a.$$

Note that all ordinals are (weakly) directed:
for any $x, y < a$, there exists $z < a$ such that $x, y \leq z$.

# Directed Refinement

**Definition**

We consider full subcategory

$$\mathcal{D} := \mathbf{ZFAlg}_{\text{infl,dir}}$$

of ZF-algebras with inflationary successors in which all downclasses are closed under binary suprema:

$$\forall x, y \in \Downarrow a, \quad x \vee y \in \Downarrow a.$$

Note that all ordinals are (weakly) directed:
for any $x, y < a$, there exists $z < a$ such that $x, y \leq z$.

**Directed ordinals:** initial object of $\mathcal{D}$.

# Construction

We start from the ordinals $N = (T^2, \subseteq, \bar{s})$.

**Definition**

For any ordinals $x_1, x_2$, we define:

$$x_1 * x_2 = x_1 \cup x_2 \cup \{y_1 * y_2 : y_1 \in x_1, y_2 \in x_2\}.$$

# Construction

We start from the ordinals $N = (T^2, \subseteq, \bar{s})$.

**Definition**

For any ordinals $x_1, x_2$, we define:

$$x_1 * x_2 = x_1 \cup x_2 \cup \{y_1 * y_2 : y_1 \in x_1, y_2 \in x_2\}.$$

**Lemma**

*If $x_1, x_2$ are hereditarily transitive sets, then so is $x_1 * x_2$.*

## Hereditarily Directedness

Idempotence ($x = x * x$) does not hold in general:

$$x * x = x \cup \{y_1 * y_2 : y_1, y_2 \in x\}.$$

# Hereditarily Directedness

Idempotence ($x = x * x$) does not hold in general:

$$x * x = x \cup \{y_1 * y_2 : y_1, y_2 \in x\}.$$

**Definition**

An ordinal $x$ is called **directed** if for all $y_1, y_2 \in x$ we have $y_1 * y_2 \in x$. An ordinal $x$ is called **hereditarily directed** if $x$ is directed and any $y \in x$ is directed as well.

# The resulting ZF-algebra

**Proposition**

*Let $G$ be the class of hereditarily directed ordinals.*

- *Order:* $\subseteq$,
- *Successor:* $\bar{s}(x) = x \cup \{x\}$,
- *Suprema: for* $Y \subseteq G$*, first close* $Y$ *under* $*$*, then take the union.*

# The resulting ZF-algebra

**Proposition**

*Let $G$ be the class of hereditarily directed ordinals.*

- *Order:* $\subseteq$,
- *Successor:* $\bar{s}(x) = x \cup \{x\}$,
- *Suprema: for* $Y \subseteq G$, *first close* $Y$ *under* $*$, *then take the union.*

*Then*

$$J = (G, \subseteq, \bar{s})$$

*is a ZF-algebra with inflationary successor. Moreover, for every* $x \in G$, *the strict downclass* $\Downarrow x$ *is closed under binary suprema.*

# Main Theorem

**Theorem**

*$J = (G, \subseteq, \bar{s})$ is the* ***initial*** *ZF-algebra with:*

- *inflationary successor,*
- *downclasses closed under binary suprema.*

So we found our directed ordinals!

# Relation to previous work

**Paul Taylor (1996):** Introduced an equivalent notion of directed ordinal with inflationary successor and characterized as free structure. However, he argued in favour of Tarski ordinals instead.

# Relation to previous work

**Paul Taylor (1996):** Introduced an equivalent notion of directed ordinal with inflationary successor and characterized as free structure. However, he argued in favour of Tarski ordinals instead.

**This work:**

- Predicative framework
- Independent development in Algebraic Set Theory
- Proof of the initial/freeness characterization

# Conclusion

**Result**

Constructive and predicative directed ordinals.

- Inflationary successor (as in CZF and HoTT)
- Directed (as in Tarski ordinals)

Classically: coincides with usual notion of ordinal.

# Bibliography I

Aczel, Peter, and Michael Rathjen. 2001. *Notes on constructive set theory.*

Bauer, Andrej, and Peter LeFanu Lumsdaine. 2013. "On the Bourbaki–Witt principle in toposes." In *Mathematical Proceedings of the Cambridge Philosophical Society,* 155:87–99. 1. Cambridge University Press.

Grayson, Robin J. 2006. "Heyting-valued models for intuitionistic set theory." In *Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis, Durham, July 9–21, 1977,* 402–414. Springer.

# Bibliography II

Joyal, André, and Ieke Moerdijk. 1995. *Algebraic Set Theory.* Vol. 220, London Mathematical Society Lecture Note Series. Cambridge University Press.

Powell, William C. 1975. "Extending Gödel's negative interpretation to ZF." *The Journal of Symbolic Logic* 40 (2): 221–229.

Taylor, Paul. 1996. "Intuitionistic sets and ordinals." *The journal of symbolic logic* 61 (3): 705–744.

The Univalent Foundations Program. 2013. *Homotopy Type Theory: Univalent Foundations of Mathematics.* Https://homotopytypetheory.org/book. Institute for Advanced Study.

Thank you :)