The lattice of smooth sublocales as a Bruns–Lakser completion

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Igor Arrieta (joint work with Anna Laura Suarez)
University of the Basque Country UPV/EHU

EHU

Background and notation

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$$\bigvee_{i\in I} a_i \wedge b = (\bigvee_{i\in I} a_i) \wedge b$$

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$$\begin{tikzcd}
\mathbf{Top} \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathbf{Loc} \arrow[l, bend left, "\Sigma"] \\
\mathbf{Sob} \arrow[u, hook] \arrow[r, "\cong"'] & \mathbf{SpLoc} \arrow[u, hook]
\end{tikzcd}$$

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- In the system $\mathsf{S}(L)$ of all sublocales of L, partially ordered by inclusion, infima and suprema are given by

$$\bigwedge_{i\in I} S_i = \bigcap_{i\in I} S_i, \quad \bigvee_{i\in I} S_i = \{ \bigwedge M \mid M \subseteq \bigcup_{i\in I} S_i \}.$$

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- For any $a \in L$, the sublocales

$$\mathfrak{c}(a) = {\uparrow}a = \{\, b \in L \mid b \geq a \,\} \text{ and } \mathfrak{o}(a) = \{\, a \to b \mid b \in L \,\}$$

  are the **closed** and **open** sublocales of L, respectively, and $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other in $\mathsf{S}(L)$ and satisfy

$$\bigcap \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee a_i), \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b),$$

$$\bigvee \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee a_i) \quad \text{and} \quad \mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b).$$

Given a locale L, the system $\mathsf{S}(L)$ is a coframe (the order-theoretic dual of a frame) and it is zero-dimensional, in the sense that

$$\mathsf{S}(L) = \left\{ \bigcap_{a \in A,\, b \in B} \mathfrak{o}(a) \vee \mathfrak{c}(b) \mid A, B \subseteq L \right\}.$$

The Bruns–Lakser completion of a join-semilattice

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Let S be a join-semilattice (the original construction is described for meet-semilattices, and the completion consists of the downsets closed under admissible joins!) A family $\{a_i\}_{i \in I} \subseteq S$ is **admissible** if its meet in S exists and for each $b \in S$, the equality

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A nonempty upper set U of S is **admissible** if it is closed under admissible meets — i.e., whenever $\{a_i\}_{i\in I} \subseteq U$ is admissible, one has $\bigwedge_{i\in I} a_i \in U$.

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We call the natural antitone embedding $\uparrow : S \to \mathcal{AU}(S)$ the **Bruns–Lakser completion** of S.

Moreover, we say that a map $f : S \to T$ between join-semilattices **lifts** if there is a frame homomorphism $\hat{f} : \mathcal{AU}(S) \to \mathcal{AU}(T)$ such that the diagram

$$\begin{array}{ccc} \mathcal{AU}(S) & \xrightarrow{\hat{f}} & \mathcal{AU}(T) \\ \uparrow & & \uparrow \\ S & \xrightarrow{f} & T \end{array}$$

commutes.

The system of joins of closed sublocales

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Let L be a frame. We write $\mathsf{S}_c(L)$ for the collection of all joins of closed sublocales of L, namely

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We consider $\mathsf{S}_c(L)$ as an ordered set with the inclusion order inherited from $\mathsf{S}(L)$.

Theorem (Picado, Pultr, Tozzi, 2019)

For any frame L, the ordered set $\mathsf{S}_c(L)$ is a frame, embedded as a complete join-sublattice of the coframe $\mathsf{S}(L)$.

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For large classes of frames (subfit frames, frames which are coframes, T_D-spatial frames,...) $\mathsf{S}_c(L)$ is a coframe. However, we recently showed that it is not always a coframe:

- IA, Joins of closed sublocales are not always a coframe, *Order* **43** (2026), art. no. 15.

Theorem

There is a locale L for which $\mathsf{S}_c(L)$ is not a coframe.

$\mathsf{S}_c(L)$ as a Bruns–Lakser completion

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$S_c(L)$ as a Bruns–Lakser completion

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Theorem (Ball, Moshier, Pultr, 2020)

Let L be a frame.

1. *A family $\{a_i\}_{i\in I}$ of L (qua join-semilattice) is* ***admissible*** *if and only if it is* ***exact*** *(i.e., the join $\bigvee_{i\in I} \mathfrak{c}(a_i)$ is closed).*

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2. *The Bruns–Lakser completion of L (qua join-semilattice) is isomorphic to $\mathsf{S}_c(L)$.*

The proof is based on the following diagram:

$$\begin{tikzcd}
\mathsf{S}_c(L) \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\top"] \arrow[d, equal] & \mathcal{U}(L) \arrow[l, bend left, "\psi"] \\
\mathsf{S}_c(L) \arrow[r, "\cong"'] & \mathcal{AU}(L) \arrow[u, hook]
\end{tikzcd}$$

where $\varphi(S) := \{a \in L \mid \mathfrak{c}(a) \subseteq S\}$ and $\psi(U) = \bigvee_{a \in U} \mathfrak{c}(a)$.

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Examples

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Open problem. What are the "exact sublocales" (namely those whose surjection S_c-lifts)?

- A. L. Suarez, Raney extensions: A pointfree theory of T_0 spaces based on canonical extension, *Journal of Pure and Applied Algebra* **229** (2025), Article no. 108137.

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Theorem (Suarez 2025)

A frame homomorphism $f\colon L \to M$ S_c-lifts if and only if it is exact: f sends exact families into exact families and it preserves their meets.

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We denote by $\mathsf{S}_b(L)$ the subset of $\mathsf{S}(L)$ consisting of **smooth sublocales**, i.e.

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Theorem (Picado, Pultr, Tozzi, 2019)

The following are equivalent for a locale L:

1. *L is subfit,*
2. *$\mathsf{S}_c(L)$ is Boolean,*
3. *$\mathsf{S}_c(L) = \mathsf{S}_b(L)$.*

$S_b(L)$ has attracted attention in point-free topology, for instance:

- As a maximal essential extension in the category of frames
  - R. N. Ball and A. Pultr, Maximal essential extensions in the context of frames, *Algebra Universalis* **79** (2018), Art. no. 32.
- As the T_D-hull of a frame
  - G. Bezhanishvili, R. Raviprakash, A. L. Suarez, and J. Walters-Wayland, The Funayama envelope as the TD-hull of a frame, *Theory and Applications of Categories* **44** (2025), Art. no. 33, 1106–1147.
- As a tool for modeling discontinuous locale maps:
  - J. Picado and A. Pultr, A Boolean extension of a frame and a representation of discontinuity, *Quaestiones Mathematicae* **40** (2017), 1111–1125.

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- IA, On joins of complemented sublocales, *Algebra Universalis* **83** (2022), Article no. 1.

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The join-semilattice of locally closed sublocales

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For a frame L, we define

$$\mathbf{LC}(L) := \{(a, b) \in L \times L \mid a \leq b,\ b \to a = a\}$$

and endow it with the following ordering:

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3. $\mathbf{LC}(L)$ *is anti-isomorphic to the subcollection of locally closed sublocales of L.*

Admissible meets in $\mathbf{LC}(L)$

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Proposition

Let $\{(a_i, b_i)\}_{i\in I} \subseteq \mathbf{LC}(L)$. The meet $\bigsqcap_{i\in I}(a_i, b_i)$ is admissible if and only if the sublocale

$$\bigvee_{i\in I} \mathfrak{c}(a_i) \cap \mathfrak{o}(b_i)$$

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Example

A family of the form $\{(a_i, 1)\}_{i\in I}$ is locally exact if and only if

$$a_i \vee \bigwedge_{x\in L}[(\bigwedge_i(x \vee a_i)) \to (x \vee \bigwedge_j a_j)] = 1, \quad \text{for all } i \in I.$$

There is a monotone map

$$\varphi\colon \mathsf{S}_b(L) \longrightarrow \mathcal{U}(\mathbf{LC}(L))$$

to the nonempty upper sets of $\mathbf{LC}(L)$, given by

$$\varphi(S) = \{(a,b) \in \mathbf{LC}(L) \mid \mathfrak{c}(a) \cap \mathfrak{o}(b) \subseteq S\}.$$

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Moreover, there is another monotone map

$$\psi\colon \mathcal{U}(\mathbf{LC}(L)) \to \mathsf{S}_b(L)$$

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Proposition

The relation $\varphi(\psi(U)) = U$ holds if and only if U is admissible.

Corollary

There is an isomorphism

$$\mathsf{S}_b(L) \cong \mathcal{A}\mathcal{U}(\mathbf{LC}(L)).$$

$$\begin{tikzcd}
\mathsf{S}_b(L) \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\top"] \arrow[d, equal] & \mathcal{U}(\mathbf{LC}(L)) \arrow[l, bend left, "\psi"] \\
\mathsf{S}_b(L) \arrow[r, "\cong"'] & \mathcal{A}\mathcal{U}(\mathbf{LC}(L)) \arrow[u, hook]
\end{tikzcd}$$

Lifting morphisms to $\mathsf{S}_b(L)$

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Recall that a map $f\colon S \to T$ between join-semilattices **lifts** if there is a frame homomorphism $\hat{f}\colon \mathcal{AU}(S) \to \mathcal{AU}(T)$ such that the diagram

$$\begin{array}{ccc} \mathcal{AU}(S) & \xrightarrow{\hat{f}} & \mathcal{AU}(T) \\ \uparrow & & \uparrow \\ S & \xrightarrow{\quad f \quad} & T \end{array}$$

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commutes.

Theorem

If $f : S \to T$ is a map between join-semilattices, then it lifts to a frame map $\mathcal{AU}(S) \to \mathcal{AU}(T)$ if and only if f is ***admissible*** *(i.e., it preserves admissible families and their meets).*

A frame homomorphism $f\colon L \to M$ will be said to be **locally exact** if the map $\mathbf{LC}(f)\colon \mathbf{LC}(L) \to \mathbf{LC}(M)$ given by $\mathbf{LC}(f)(a,b) = lc(f(a), f(b))$ is admissible.

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$$\prod_{i\in I} lc(f(a_i), f(b_i)) = lc(f \times f)(\prod_{i\in I}(a_i, b_i)).$$

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Corollary

A frame homomorphism $f : L \to M$ S_b-lifts if and only if it is locally exact.

Thank you!

For further details, see our paper:

- IA, A. L. Suarez The lattice of smooth sublocales as a Bruns – Lakser completion, *Appl. Categ. Struct.* **34** (2026), Art. no. 37.