Defeasible Reasoning on Concepts:Non-distributive defeasible logics

Y. Ding[2] K. Manoorkar[1] A. Palmigiano[2] N. W. Switrayni[2,3] R. Wang[4]

[1]Institute of Computer Science, Czech Academy of Sciences
[2]Vrije Universiteit Amsterdam
[3]University of Mataram, Indonesia
[4]Northwestern University, China

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Motivation

KLM (Kraus, Lehmann, and Magidor, 1990) Framework

For defeasible reasoning:

$$\alpha \mid\!\sim \beta\text{: if } \alpha\text{, normally } \beta\text{, or } \beta \text{ is a plausible consequence of } \alpha.$$

Note: The relation $\mid\!\sim$ is inherently non-monotonic.

$$\begin{aligned} x \text{ is a bird} &\mid\!\sim x \text{ can fly} \\ x \text{ is a penguin} &\mid\!\sim x \text{ is a bird} \\ x \text{ is a penguin} &\not\mid\!\sim x \text{ can fly} \end{aligned}$$

KLM Framework for Defeasible Reasoning

The language L of cumulative logic consists of:

- Logical connectives: $\vee, \neg, \wedge, \rightarrow, \leftrightarrow$
- Propositional variables

Thus, L is the set of standard propositional formulas.

Cumulative Logical System

Ref $\phi \mid\!\sim \phi$

RW $$\frac{\phi \rightarrow \psi \quad \chi \mid\!\sim \phi}{\chi \mid\!\sim \psi}$$

LLE $$\frac{\phi \leftrightarrow \psi \quad \phi \mid\!\sim \chi}{\psi \mid\!\sim \chi}$$

Cut $$\frac{\phi \mid\!\sim \psi \quad \phi \wedge \psi \mid\!\sim \chi}{\phi \mid\!\sim \chi}$$

CM $$\frac{\phi \mid\!\sim \psi \quad \phi \mid\!\sim \chi}{\phi \wedge \psi \mid\!\sim \chi}$$

Cumulative Models

Definition (Cumulative Model)

A structure $\mathcal{W} = (S, l, \prec)$ where:

- S is a set of states,
- $l : S \to \mathcal{P}(\mathcal{U})$ assigns possible worlds to states,
- $\prec$ is a preference relation on states,

is cumulative if, for every ϕ, the set $\widehat{\phi} = \{s \in S \mid s \models \phi\}$ is **smooth**.

Smoothness Condition

Every state satisfying ϕ has a minimal state below it (or is itself minimal).

Cumulative Consequence Relations

Consequence relation induced by a model

A cumulative model $\mathcal{W}$ defines:

$$\phi \mid\sim_{\mathcal{W}} \psi \quad \Longleftrightarrow \quad \min(\widehat{\phi}) \subseteq \widehat{\psi}$$

Representation Theorem:

Cumulative relations are exactly those induced by cumulative models.

Additional Rules

$$\frac{\phi_0 \mid\sim \phi_1 \quad \cdots \quad \phi_{n-1} \mid\sim \phi_n \qquad \phi_n \mid\sim \phi_0}{\phi_0 \mid\sim \phi_n} \text{ (Loop)}$$

$$\frac{\phi \mid\sim \chi \qquad \psi \mid\sim \chi}{\phi \vee \psi \mid\sim \chi} \text{ (Or)}$$

Systems Overview

System	Models
C	Cumulative
CL $=$ **C** $+$ Loop	Ordered cumulative
P $=$ **C** $+$ Or	Preferential

Polarity-based Semantics

Definition (Polarity)

A **polarity** is a tuple $\mathbb{P} = (A, X, I)$, where:

- A is a set of objects,
- X is a set of features,
- $I \subseteq A \times X$ is the object–feature incidence relation.

Galois Connection

The maps:

$$B^{\uparrow} := \{x \in X \mid \forall b \in B,\ bIx\}, \qquad Y^{\downarrow} := \{a \in A \mid \forall y \in Y,\ aIy\}$$

form a Galois connection, meaning:

$$Y \subseteq B^{\uparrow} \iff B \subseteq Y^{\downarrow}$$

Formal Concepts & Concept Lattices

Definition (Formal Concept)

A **formal concept** (or category) of $\mathbb{P}$ is a pair $c = ([\![c]\!], (\![c]\!))$ such that $[\![c]\!] \subseteq A$, $(\![c]\!) \subseteq X$, and:

$$[\![c]\!]^{\uparrow} = (\![c]\!) \quad \text{and} \quad (\![c]\!)^{\downarrow} = [\![c]\!]$$

Thus, $[\![c]\!]$ and $(\![c]\!)$ are *Galois-stable* ($[\![c]\!]^{\uparrow\downarrow} = [\![c]\!]$ and $(\![c]\!)^{\downarrow\uparrow} = (\![c]\!)$).

Definition (Concept Lattice)

The concept lattice $\mathbb{P}^{+}$ consists of all formal concepts of $\mathbb{P}$, partially ordered by:

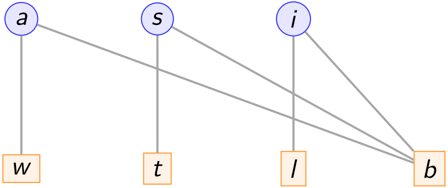
$$c \leq d \quad \Longleftrightarrow \quad [\![c]\!] \subseteq [\![d]\!] \quad \Longleftrightarrow \quad (\![d]\!) \subseteq (\![c]\!)$$

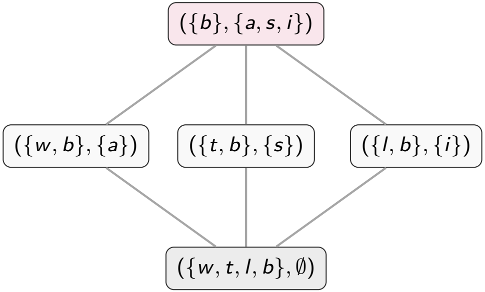
where the meet ($\wedge$) and join ($\vee$) are given by:

$$\bigwedge \mathcal{H} := \left(\bigcap_{c \in \mathcal{H}} [\![c]\!],\ \left(\bigcap_{c \in \mathcal{H}} [\![c]\!]\right)^{\uparrow} \right)$$

Formal Context & Lattice: Consumer Electronics

Attributes: a = Audio, s = Screen, i = Internet
Objects: w = Walkman, t = TV, l = LAN Router, b = BlackBerry





Lattice-based Propositional Logic **L**

The language $\mathcal{L}$ of **L** is defined recursively:

$$\phi ::= p \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi$$

Axioms

L is the smallest logic containing:

$$p \vdash p \qquad p \vdash \top \qquad \bot \vdash p$$

$$p \vdash p \vee q \qquad q \vdash p \vee q \qquad p \wedge q \vdash p \qquad p \wedge q \vdash q$$

Inference Rules

Closed under the following rules:

$$\frac{\phi \vdash \chi \quad \chi \vdash \psi}{\phi \vdash \psi} \qquad \frac{\phi \vdash \psi}{\phi(\chi/p) \vdash \psi(\chi/p)}$$

$$\frac{\chi \vdash \phi \quad \chi \vdash \psi}{\chi \vdash \phi \wedge \psi} \qquad \frac{\phi \vdash \chi \quad \psi \vdash \chi}{\phi \vee \psi \vdash \chi}$$

Definition (Polarity-based Model)

A pair $\mathbb{M} = (\mathbb{P}, V)$, where $\mathbb{P}$ is a polarity, and $V : \mathsf{Prop} \to \mathbb{P}^+$ is a valuation assigning a concept to each propositional variable.
Extensions/intensions are denoted as:

$$[\![p]\!] := [\![V(p)]\!] \qquad (\![p]\!) := (\![V(p)]\!)$$

Satisfaction ($\Vdash$) and Co-satisfaction ($\succ$):

$\mathbb{M}, a \Vdash p$	iff $a \in [\![p]\!]_{\mathbb{M}}$	$\mathbb{M}, x \succ p$	iff $x \in (\![p]\!)_{\mathbb{M}}$
$\mathbb{M}, a \Vdash \top$	always	$\mathbb{M}, x \succ \top$	iff $(\forall a \in A)\ aIx$
$\mathbb{M}, a \Vdash \bot$	iff $(\forall x \in X)\ aIx$	$\mathbb{M}, x \succ \bot$	always
$\mathbb{M}, a \Vdash \phi \wedge \psi$	iff $\mathbb{M}, a \Vdash \phi$ and $\mathbb{M}, a \Vdash \psi$	$\mathbb{M}, x \succ \phi \wedge \psi$	iff $(\forall a \in A)(\mathbb{M}, a \Vdash \phi \wedge \psi \Rightarrow aIx)$

Polarity-based Models (Cont.)

Disjunction Semantics

$$\mathbb{M}, x \succ \phi \vee \psi \iff \mathbb{M}, x \succ \phi \text{ and } \mathbb{M}, x \succ \psi$$
$$\mathbb{M}, a \Vdash \phi \vee \psi \iff \forall x \in X \, (\mathbb{M}, x \succ \phi \vee \psi \Rightarrow aIx)$$

Validity

For any $\phi, \psi \in \mathcal{L}$:

$$\mathbb{M}, a \Vdash \phi \iff a \in [\![\overline{V}(\phi)]\!] \qquad \mathbb{M}, x \succ \phi \iff x \in (\![\overline{V}(\phi)]\!)$$

Consequence holds iff extensions/intensions match:

$$\mathbb{M} \models \phi \vdash \psi \iff [\![\phi]\!] \subseteq [\![\psi]\!] \iff (\![\psi]\!) \subseteq (\![\phi]\!)$$

Non-Monotonic Consequence Relations in FCA

Relation	**Interpretation**	**Real-World Example**
$C_1 \vdash C_2$	Strict classical inclusion ($C_1 \subseteq C_2$)	Penguins $\vdash$ Birds
$C_1 \mid\sim_A C_2$	Typical C_1 objects are in C_2	Mammals $\mid\sim_A$ Viviparous
$C_1 \mid\sim_X C_2$	C_1 objects have typical features of C_2	Smartwatches $\mid\sim_X$ Wrist-watches
$C_1 \mid\sim_{AX} C_2$	Typical C_1 objects have typical features of C_2	Urban Parks $\mid\sim_{AX}$ Natural Forests

Defeasibility Example ($\mid\sim_A$)

- *Default rule:* $C_1 \mid\sim_A C_2$ (Mammals $\mid\sim_A$ Viviparous) holds.
- *Defeated context:* If $C_3 =$ Echidnas, then $C_3 \wedge C_1 \not\mid\sim_A C_2$.

Objective: Formalize $C_1 \mid\sim_A C_2$, $C_1 \mid\sim_X C_2$, and $C_1 \mid\sim_{AX} C_2$ within the KLM framework.

Conceptual Cumulative Model

Definition ((Dual) Conceptual Cumulative Model)

A structure $\mathbb{W} = (S, L, \prec)$ where:

- S is a set of states,
- $L : S \to \mathcal{P}(\mathcal{M})$ assigns each state a non-empty set of (dual) pointed polarity-based models,
- $\prec$ is a preference relation on S,

is called a **(Dual) Conceptual Cumulative Model** if, for every ϕ, the set $\widehat{\phi} = \{s \in S \mid s \models \phi\}$ is smooth.

Definition (Conceptual Bi-Cumulative Model)

A pair $\mathcal{B} = (\mathcal{M}_A, \mathcal{N}_X)$ where $\mathcal{M}_A$ and $\mathcal{N}_X$ are (dual) conceptual cumulative models respectively.

Ordered and Preferential Models

Conceptual Cumulative Ordered Model

A (dual) conceptual cumulative model $\mathcal{M} = (S, l, \prec)$ is a **(dual) conceptual cumulative ordered model** if $\prec$ is a strict partial order.

Conceptual Preferential Model

A (dual) conceptual cumulative model $\mathcal{M} = (S, l, \prec)$ is a **(dual) conceptual preferential model** if the label l assigns a single pointed polarity-based model to each state.

Conceptual Cumulative Relations

Conceptual Cumulative Relation

For a (dual) conceptual cumulative model $\mathcal{M} = (S, I, \prec)$:

$$\phi_1 \mid\!\sim_{\mathcal{M}} \phi_2 \quad \Longleftrightarrow \quad \min(\widehat{\phi_1}) \subseteq \widehat{\phi_2}$$

Conceptual Bi-Cumulative Consequence

A relation $\mid\!\sim_{\mathcal{B}_{AX}}$ is defined by $\phi \mid\!\sim_{\mathcal{B}_{AX}} \psi$ iff:

Every minimal ϕ-state on the object side and every minimal ψ-state on the feature side induce related points:

$$\mathbb{M}_a, \mathbb{N}_x \text{ from the same polarity} \quad \Rightarrow \quad aIx$$

Conceptual Cumulative Systems

CC System

$$\phi \mathrel{|\!\!\sim_A} \phi$$

$$\frac{\phi \vdash \psi \quad \psi \vdash \phi \quad \phi \mathrel{|\!\!\sim_A} \chi}{\psi \mathrel{|\!\!\sim_A} \chi}$$

$$\frac{\phi \vdash \psi \quad \chi \mathrel{|\!\!\sim_A} \phi}{\chi \mathrel{|\!\!\sim_A} \psi}$$

$$\frac{\phi \mathrel{|\!\!\sim_A} \psi \quad \phi \mathrel{|\!\!\sim_A} \chi}{\phi \wedge \psi \mathrel{|\!\!\sim_A} \chi}$$

$$\frac{\phi \wedge \psi \mathrel{|\!\!\sim_A} \chi \quad \phi \mathrel{|\!\!\sim_A} \psi}{\phi \mathrel{|\!\!\sim_A} \chi}$$

CC^{op} System

$$\phi \mathrel{|\!\!\sim_X} \phi$$

$$\frac{\phi \vdash \psi \quad \psi \vdash \phi \quad \chi \mathrel{|\!\!\sim_X} \phi}{\chi \mathrel{|\!\!\sim_X} \psi}$$

$$\frac{\phi \vdash \psi \quad \psi \mathrel{|\!\!\sim_X} \chi}{\phi \mathrel{|\!\!\sim_X} \chi}$$

$$\frac{\psi \mathrel{|\!\!\sim_X} \phi \quad \chi \mathrel{|\!\!\sim_X} \phi}{\chi \mathrel{|\!\!\sim_X} (\psi \vee \phi)}$$

$$\frac{\chi \mathrel{|\!\!\sim_X} (\psi \vee \phi) \quad \psi \mathrel{|\!\!\sim_X} \phi}{\chi \mathrel{|\!\!\sim_X} \phi}$$

Equilibrium Bi-Cumulative System

$$\mathbf{EBC} = \mathbf{CC} + \mathbf{CC}^{op}$$

Additional Interaction Rules

$$\frac{\phi \vdash \psi \quad \psi \vdash \phi \quad \phi \mathrel{|\!\!\sim}_{AX} \chi}{\psi \mathrel{|\!\!\sim}_{AX} \chi} \qquad \frac{\phi \mathrel{|\!\!\sim}_{A} \psi \quad \phi \mathrel{|\!\!\sim}_{AX} \chi}{\phi \wedge \psi \mathrel{|\!\!\sim}_{AX} \chi}$$

$$\frac{\phi \mathrel{|\!\!\sim}_{A} \psi}{\phi \mathrel{|\!\!\sim}_{AX} \psi} \qquad \frac{\psi \mathrel{|\!\!\sim}_{X} \phi}{\psi \mathrel{|\!\!\sim}_{AX} \phi} \qquad \frac{\psi \mathrel{|\!\!\sim}_{X} \phi \quad \chi \mathrel{|\!\!\sim}_{AX} \phi}{\chi \mathrel{|\!\!\sim}_{AX} (\psi \vee \phi)}$$

Representation Theorem

Definition (Normal Pointed Model)

A pointed polarity-based model $\mathbb{M}_a$ is **normal** for a concept ϕ iff for all $\psi \in \mathcal{L}$, $\phi \mathrel{|\!\sim} \psi$ implies $\mathbb{M}_a \Vdash \psi$.

Lemma

Let $\mathrel{|\!\sim}$ be a cumulative consequence relation and ϕ, ϕ' be any concepts. For all ϕ', $\phi \mathrel{|\!\not\sim} \phi'$ iff there exists a pointed polarity-based model $\mathbb{M}_a$ normal for ϕ, such that $\mathbb{M}_a \nVdash \phi'$.

Theorem (Representation Theorem)

A conceptual consequence relation is a cumulative consequence relation iff it is defined by some conceptual cumulative model.

Conceptual Preferential Model

Definition (Supernormal Pointed Model)

A pointed polarity-based model $\mathbb{M}_a$ is **supernormal** for a concept ϕ iff for all $\psi \in \mathcal{L}$:

$$\phi \mathrel{|\!\sim} \psi \iff \mathbb{M}_a \Vdash \psi$$

Lemma

For any $\phi \in \mathcal{L}$, there exists a pointed polarity-based model $\mathbb{M}_a$ which is supernormal for ϕ.

Conceptual vs. Classical Setting

- **Rule (Or) Invalidity:** In lattice-based logic, it is possible that $\mathbb{M}, a \Vdash \phi \vee \psi$, but $\mathbb{M}, a \not\Vdash \phi$ and $\mathbb{M}, a \not\Vdash \psi$. Unlike classical preferential models, $\widehat{\phi \vee \psi} = \widehat{\phi} \cup \widehat{\psi}$ is not valid in conceptual preferential models. Thus, the rule **(Or)** fails.
- **Rule (Loop):** In the propositional setting, the rule **(Loop)** is valid on all preferential models. However, this is not the case in the conceptual setting.
- **Supernormality:** Conceptually requires stronger structural properties than classical typicality mappings.

Conclusion and Future Work

Conclusions

- We successfully generalized the KLM defeasible reasoning framework to a conceptual setting using Formal Concept Analysis (FCA).
- Provided a sound and complete axiomatization for reasoning over concept lattices.

Future Work

- Axiomatic extensions (e.g., rational monotonicity).
- Typicality
- Complexity.
- Belief revision