Coequivalence relations and descent in modal logic

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Coequivalence relations

Definability

We are interested in the following family of problems in logic:

What implicitly defined objects can a logic explicitly define?

Beth definability property: can implicit functions be made explicit?

Definition

A logic L has the Beth definability property if for any $\varphi(\overline{p}, q)$:

$$\text{if } \varphi(\overline{p}, q) \wedge \varphi(\overline{p}, q') \vdash_L q \leftrightarrow q' \ ,$$

$$\text{then there exists } \psi(\overline{p}) \text{ s.t. } \varphi(\overline{p}, q) \vdash_L q \leftrightarrow \psi(\overline{p}) \ .$$

Uniform interpolation: explicitly defining propositional quantifiers.

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A definability problem first studied by Ghilardi and Zawadowski (2002):

Given a "provable equivalence relation" of the logic, can the logic explicitly define the equivalence classes?

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- (Context) $\rho(\overline{p}_1, \overline{p}_2) \vdash_L \tau(\overline{p}_1) \wedge \tau(\overline{p}_2)$;
- (Reflexivity) $\tau(\overline{p}) \vdash_L \rho(\overline{p}, \overline{p})$;
- (Symmetry) $\rho(\overline{p}_1, \overline{p}_2) \vdash_L \rho(\overline{p}_2, \overline{p}_1)$;
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Separation of variables

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A τ-coequivalence relation $\rho(\overline{p}_1, \overline{p}_2)$ is said to separate variables if there are $\psi_1(\overline{p}), \ldots, \psi_n(\overline{p})$ such that

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Definition

We say that a logic L has the coequivalence separation property (CoSP) if every coequivalence relation separates variables.

Model theoretic interpretation: an instance of elimination of imaginaries[1].

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Failure of the CoSP

CPC enjoys the CoSP: a model w over $\overline{p}_1$ and $\overline{p}_2$ is uniquely determined by the restrictions $w{\restriction}_{\overline{p}_1}$ and $w{\restriction}_{\overline{p}_2}$.[2]

The CoSP fails in IPC [GZ2002]: $\rho(p_1, p_2) \coloneqq \neg\neg(p_1 \wedge p_2) \vee (p_1 \leftrightarrow p_2)$

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$\mathfrak{M}{\restriction}_{p_1} \simeq \mathfrak{M}{\restriction}_{p_2}$

The CoSP fails also in KC, K, K4, S4.

[2] This resembles a sheaf-like condition, similar to the uniqueness of gluing.

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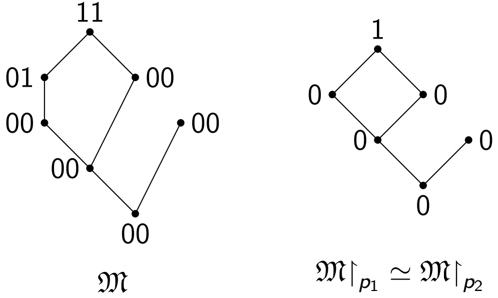

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C with finite limits. $X \otimes Y$ product of X and Y in **C**.

Definition

An equivalence relation over X is a mono $(p_1, p_2)\colon R \longrightarrow X \otimes X$:

- (R) $r\colon X \longrightarrow R$ s.t. $p_1 r = \mathrm{id}_X$ and $p_2 r = \mathrm{id}_X$;
- (S) $s\colon R \longrightarrow R$ s.t. $p_1 s = p_2$ and $p_2 s = p_1$;
- (T) $t\colon R \otimes_X R \longrightarrow R$ s.t. $p_1 t = p_1 q_1$ and $p_2 t = p_2 q_2$:

$$\begin{tikzcd} R \otimes_X R \arrow[r, "q_2"] \arrow[d, "q_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "p_1"] \\ R \arrow[r, "p_2"'] & X \end{tikzcd} .$$

Categorical interpretation

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Effective equivalence relations

Effectiveness: "pick equivalence classes with a map".

Definition

An equivalence relation $(p_1, p_2)\colon R \longrightarrow X \otimes X$ is said to be effective if there is $f\colon X \longrightarrow Y$ such that

$$\begin{tikzcd} R \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}.$$

CoSP, algebras and duals

For normal modal logics (or superintuitionistic logics), then:

Theorem

L has the CoSP if(f) every equivalence relation in $\mathrm{Alg}(L)^{\mathrm{op}}_{\mathrm{fp}}$ is effective.[a]

[a] Together with regularity, this property is called Barr-exactness.

Coequivalence separation in S5

The S5 case

S5-algebras are Boolean algebras A, with $\Box\colon A \longrightarrow A$ s.t. ($\Diamond :=\sim \Box \sim$)

$$\Box 1 = 1; \qquad \Box(a \wedge b) = \Box a \wedge \Box b; \qquad \Box a \leq a;$$

$$\Box a \leq \Box\Box a; \qquad \Diamond a \leq \Box\Diamond a.$$

We have $\mathrm{Alg}(\mathrm{S5})^{\mathrm{op}}_{\mathrm{fp}}$:[3]

- **Objects**: finite families $\mathcal{X}$ of non-empty finite sets;
- **Arrows** $\mathcal{X} \longrightarrow \mathcal{Y}$: functions $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ with a family of surjective functions $f_X\colon X \longrightarrow f(X)$, for each $X \in \mathcal{X}$.

[3] This representation goes back to the work of Horn (1978) and Bass (1958).

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Bisimulations

Definition

A bisimulation from $X \neq \emptyset$ to $Y \neq \emptyset$ is $A \subseteq X \times Y$:

- $\forall x \in X$, $\exists y \in Y$ s.t. $(x, y) \in A$;
- $\forall y \in Y$, $\exists x \in X$ s.t. $(x, y) \in A$.

$\mathrm{Bis}(X, Y)$ is the set of bisimulations from X to Y.

Definition

For $\mathcal{X} \in \mathrm{Alg}(\mathrm{S5})^{\mathrm{op}}_{\mathrm{fp}}$,

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Equivalence relations in $\mathsf{Alg}(\mathsf{S5})^{\mathrm{op}}_{\mathrm{fp}}$.

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$\mathcal{R} \subseteq \mathrm{Bis}(\mathcal{X})$ is equivalential if:

- (Identity) for $X \in \mathcal{X}$,

$$\Delta_X \coloneqq \{(x,x) \mid x \in X\} \in \mathcal{R} \;;$$

- (Dagger) for $X_1, X_2 \in \mathcal{X}$ and $A \in \mathrm{Bis}(X_1, X_2)$,

$$A \in \mathcal{R} \implies A^\dagger \coloneqq \{(x_2, x_1) \mid (x_1, x_2) \in A\} \in \mathcal{R} \;;$$

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$$A, B \in \mathcal{R} \implies \mathrm{Comp}(P) \coloneqq \{(x_1, x_3) \mid ((x_1, x_2), (x_2, x_3)) \in P\} \in \mathcal{R} \;.$$

[a] $((x_1, x_2), (x_2', x_3)) \in P \implies x_2 = x_2'$.

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Failure of CoSP in S5

Proposition

Equivalential $\mathcal{R} \subseteq \mathsf{Bis}(\mathcal{X})$ *correspond to coequivalence relations for* S5.

$\mathcal{R}$ *separates variables iff, for each* $X_1, X_2 \in \mathcal{X}$ *and* $A, B \in \mathsf{Bis}(X_1, X_2)$*,*

- (D) $A \in \mathcal{R}$ *and* $B \subseteq A \implies B \in \mathcal{R}$*;*
- (U) $A, B \in \mathcal{R} \implies A \cup B \in \mathcal{R}$*.*

Theorem [R. N. Almeida, M. D.]

The CoSP *fails in* S5.

Proof. Take $X = \{1,2\}$ and $\mathcal{X} = \{X\}$. Consider

$$A := \{(1,2),(2,1)\} \in \mathsf{Bis}(X,X) \ .$$

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Descent and local coequivalences

The road we take

Due to the failure of CoSP in IPC, Ghilardi and Zawadowski (2002):

Does IPC satisfy a weaker definability property concerning "separation of two sets of independent variables"?[4]

[4] As the CoSP is related to Barr-exactness, they suggested this was related to "all regular epis being of effective descent" (implied by, but not implying, Barr-exactness).

Local transition terms and coequivalence relations

Let $\beta(\overline{p},\overline{q}) \vdash_L \alpha(\overline{p})$ and $\gamma(\overline{p},\overline{q},\overline{r}) \vdash_L \beta(\overline{p},\overline{q})$.

Definition

$(\xi_r(\overline{p},\overline{q}_1,\overline{q}_2,\overline{r})\colon r \in \overline{r})$, is a sequence of local transition terms if:

- (γ-compatibility): $\gamma(\overline{p},\overline{q}_1,\overline{r}) \wedge \beta(\overline{p},\overline{q}_2) \vdash_L \gamma(\overline{p},\overline{q}_2,\overline{\xi})$;
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Intuition: $\overline{q}$ and $\overline{r}$ are sets of independent variables; the local transition moves $\overline{r}$, consistently with γ, from some $\overline{q}_1$ to some $\overline{q}_2$.

Local transition terms and coequivalence relations

Let $\beta(\overline{p}, \overline{q}) \vdash_L \alpha(\overline{p})$ and $\gamma(\overline{p}, \overline{q}, \overline{r}) \vdash_L \beta(\overline{p}, \overline{q})$.

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A weaker property

Definition

The local γ-coequivalence relation associated is

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Intuition: local coequivalence relations indentify variables that are in the same orbit of the local transition.

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We say that L has the local coequivalence separation property (LCoSP) if every local coequivalence relation separates variables.

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Slice categories and pullbacks

C with pullbacks, $p\colon E \longrightarrow B$ in **C**.

Definition

For $B \in \mathbf{C}$, the slice category over B is $\mathbf{C}/B$:

- **Objects**: pairs (Y, g), with $g\colon Y \longrightarrow B$ in **C**;
- **Arrows** $(Y,g) \longrightarrow (Y',g')$: arrows $h\colon Y \longrightarrow Y'$ in **C** s.t. $g'h = g$.

There is an adjunction $p_! \dashv p^*\colon \mathbf{C}/B \longrightarrow \mathbf{C}/E$ s.t.

- p^* sends $(Y,g) \in \mathbf{C}/B$ to $(E \otimes_B Y, \pi_1) \in \mathbf{C}/E$, where

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Descent data

Definition

The category of descent data $\mathrm{Des}(p)$ is the category of algebras for the monad $p^* p_!$ on $\mathbf{C}/E$.

An object of $\mathrm{Des}(p)$ is $(X, f; \xi)$, where

- $(X, f) \in \mathbf{C}/E$;
- $\xi \colon E \otimes_B X \longrightarrow X$ in $\mathbf{C}$ s.t.

$$\begin{tikzcd}
X \arrow[r, "{(f,1_X)}"] \arrow[d, "1_X"'] & E \otimes_B X \arrow[d, "\pi_1"] \arrow[dl, "\xi"'] \\
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\qquad
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Effective descent morphisms

Effective descent morphisms $p\colon E \longrightarrow B$ in $\mathbf{C}$ are those morphisms

which facilitate an algebraic description of $\mathbf{C}/B$ by means of $\mathbf{C}/E$.

Definition

A morphism $p\colon E \longrightarrow B$ is of effective descent if the comparison functor $\Phi^p\colon \mathbf{C}/B \longrightarrow \mathrm{Des}(p)$ is an equivalence.

Effective descent morphisms are always regular epis.

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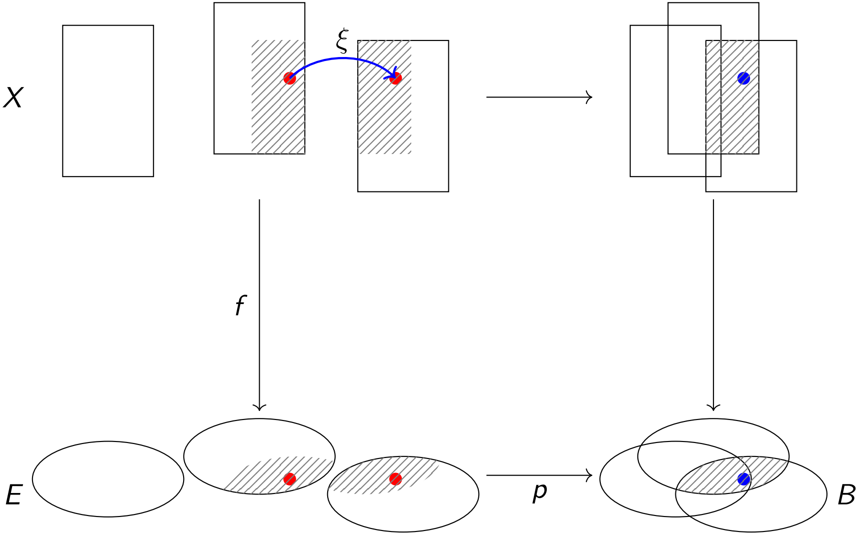
Descent pictured



For normal modal logics (or superintuitionistic logics), then:

Theorem

L has the LCoSP *iff every regular epi in* $\mathrm{Alg}(L)_{\mathrm{fp}}^{\mathrm{op}}$ *is of effective descent.*[a]

[a] Together with regularity, this property is called almost Barr-exactness.

Compatible bisimulations

Let us go back to $\mathrm{Alg}(\mathrm{S5})_{\mathrm{fp}}^{\mathrm{op}}$ and generalize $\mathrm{Bis}(\mathcal{X})$.

Definition

For $q\colon \mathcal{X} \longrightarrow \mathcal{Z}$ and $p\colon \mathcal{Y} \longrightarrow \mathcal{Z}$ in $\mathrm{Alg}(\mathrm{S5})_{\mathrm{fp}}^{\mathrm{op}}$,

$$\mathrm{Bis}(q,p) := \coprod_{\substack{X\in\mathcal{X}, Y\in\mathcal{Y}\\ q(X)=p(Y)}} \{A \in \mathrm{Bis}(X,Y) \mid (\forall (x,y)\in A)\ (q_X(x) = p_Y(y))\}$$

is the family of (q,p)-compatible bisimulations.

S5 enjoys the LCoSP

Proposition

$\mathcal{R} \subseteq \mathrm{Bis}(\mathcal{X})$ correspond to a local coequivalence relation for S5 iff there are $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ and $p\colon \mathcal{Y} \longrightarrow \mathcal{Z}$ in $\mathrm{Alg}(\mathrm{S5})^{\mathrm{op}}_{\mathrm{fp}}$ s.t.

- (Compatibility) *for each $A \in \mathcal{R}$,*

$$A_f := \{(x_1, f_{X_2}(x_2)) \mid (x_1, x_2) \in A\} \in \mathrm{Bis}(pf, p)$$

  and $A \longrightarrow A_f$ is bijective;
- (Lift) *for each $D \in \mathrm{Bis}(pf, p)$, there is a unique $A \in \mathcal{R}$ s.t. $A_f = D$.*

Theorem [R. N. Almeida, M. D.]

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Conclusions and future work

CoSP and LCoSP natural in the study of propositional non-classical logics, from a categorical, model theoretic and geometric point of view.

Future work:

- systematic approach to the study of CoSP and LCoSP;
- connections with model theory;
- connections with generalized Galois theory.

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Thanks for your attention!

CPC enjoys the CoSP

Let $\rho(\overline{p}_1, \overline{p}_2)$ be a $\top$-coequivalence relation. Valuations $v\colon \overline{p} \longrightarrow \{0,1\}$:

$$v \sim_\rho v' \iff \exists v_0\colon \overline{p}_1, \overline{p}_2 \longrightarrow \{0,1\} \text{ s.t. } \begin{cases} v_0{\restriction_{\overline{p}_1}} = v \\ v_0{\restriction_{\overline{p}_2}} = v' \\ v_0 \Vdash \rho. \end{cases}$$

$C_1, \ldots, C_n$ equivalence classes, $\psi_i(\overline{p})$ in DNF: $v \Vdash \psi_i \iff v \in C_i$.

For $w\colon \overline{p}_1, \overline{p}_2 \longrightarrow \{0,1\}$,

$$w \Vdash \bigwedge_{i=1}^{n} (\psi_i(\overline{p}_1) \leftrightarrow \psi_i(\overline{p}_2)) \iff w{\restriction_{\overline{p}_1}} \sim_\rho w{\restriction_{\overline{p}_2}}$$

$$\exists v_0 \text{ s.t. } \begin{cases} v_0{\restriction_{\overline{p}_1}} = w{\restriction_{\overline{p}_1}} \\ v_0{\restriction_{\overline{p}_2}} = w{\restriction_{\overline{p}_2}} \\ v_0 \Vdash \rho \end{cases} \iff w = v_0 \Vdash \rho.$$