On the lattice of subvarieties of equational states

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joint work in progress with
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The roadmap

1. Introduction to equational states
2. Jonsson's Lemma for equational states
3. The case for finite subdirectly irreducible equational states
4. The infinite case.

States of MV-algebra

A state is a function $s \colon A \to [0,1]$ that is normalized ($s(1_A) = 1$) and finitely additive ($s(a \oplus b) = s(a) + s(b)$ whenever $a \odot b = 0_A$).

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Kroupa-Panti theorem: the set of states of an MV-algebra A is in bijection with the set of regular Borel probability measures on the spaces of maximal ideals of A.

Equational states

Definition (Kroupa, Marra 2021)

An equational state is a two-sorted algebra $(\mathrm{A}_1, \mathrm{A}_2, s)$, where both sorts are MV-algebras and the operation s satisfy the equations

1. $s(1_{A_1}) = 1_{A_2}$,
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Note that the class of equational states forms a variety. Therefore, a natural question is which subclasses of equational states form a subvariety.

Two-sorted ideals

Definition

Let $(\mathrm{A}_1, \mathrm{A}_2, s)$ be an equational state. A Σ-ideal is a subset (J_1, J_2) such that J_i is an MV-ideal of A_i and $s(J_1) \subseteq J_2$.

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Theorem (Lapenta, N., Spada 2023)

The lattice of Σ-ideals and the lattice of Σ-congruences are isomorphic

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- A_2 *is a subdirectly irreducible MV-algebra and* s *is faithful.*

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A straightforward computation shows that $s(I_1 \vee J_1) \subseteq I_2 \vee J_2$. Therefore $\langle s(I_1 \vee J_1) \cup (I_2 \vee J_2) \rangle = I_2 \vee J_2$.

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Therefore, we can compute the lattice operations "component-wise." The following result is quite immediate, since the lattices of MV-ideals are distributive.

Theorem

The lattice of Σ-ideals is distributive for any equational state.

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Lemma

For any class $\mathcal{K}$ of equational states

$$\mathbb{V}(\mathcal{K})_{SI} \subseteq \mathbb{HSP}_U(\mathcal{K}).$$

Subdirectly irreducible finite equational states

Proposition

Let $(\mathsf{A}, Ł_k)$ *be a subdirectly irreducible equational state, with* k *a positive natural. Then* $\mathsf{A} \simeq \prod_{n=1}^{n} Ł_{k_n}$, *with* $n \in \mathbb{N}$ *and* $k_1, \ldots, k_n \in \mathbb{N} \setminus \{0\}$.

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It is easy to note that if s is faithful then $s(a) < s(b)$ whenever $a < b$. Hence A has only finite subchains. Since the lattice reduct is distributive, then A is finite.

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Lemma (Kroupa, Marra 2021)

Let $s \colon \mathrm{A} \to Ł_k$ be a function. Then s is a state if and only if

$$\sum_{i=1}^{n} k_i s(a_i) = 1.$$

Moreover, the state is uniquely determined by the value at the atoms of A.

Corollary

An equational state $(\mathrm{A}, Ł_k, s)$ *is subdirectly irreducible if and only if* $\mathrm{A} = Ł_{m_1} \times \cdots \times Ł_{m_n}$ *is finite and there exists* $m_i > 0$ *for all* $i = 1, \ldots, n$ *such that* $\sum_{i=1}^{n} k_i m_i = k$ *and* $s(a_i) = m_i/k$.

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- for 3, we derive the equational state $(Ł_3, Ł_3)$, with $s(a) = \frac{3/3}{3} = \frac{1}{3}$;
- we use a similar argument for the other partitions, using products for sums with multiple summands.

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Therefore, to see that A is a subalgebra of B is sufficient to verify that both sorts of A are subalgebras of the sorts of B and the image on a and $i_1(a)$ are the same for each atom a.

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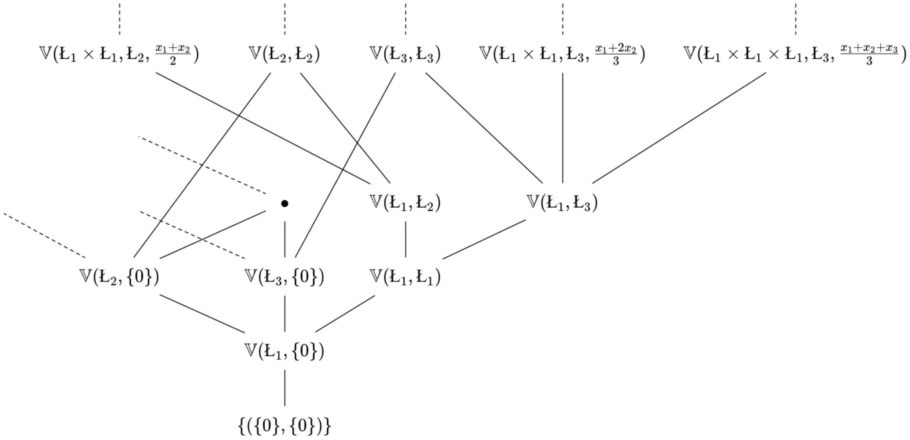
Let (A_1, A_2, s) and (B_1, B_2, t) be two subdirectly irreducible finite equational states. Then $(B_1, B_2, t) \in \mathbb{H}(A_1, A_2, s)$ if and only if $B_i \in \mathbb{H}(A_i)$ through the surjective homomorphism q_i and

$$t(q_1(a)) = q_2(s(a))$$

for any atom a of A_1.

Representing the sublattice



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Each state operation from the MV-algebra A_1 to the MV-algebra A_2 extend uniquely to a state of ℓ-groups between $\Gamma^{-1}(A_1)$ and $\Gamma^{-1}(A_2)$.

It is usually easier to work with states of ℓ-groups than to work with operation states, since $s(a+b) = s(a) + s(b)$ but usually $s(a \oplus b) \neq s(a) \oplus s(b)$.

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We assume that the first sort is finitely generated.

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Then, given a function $f\colon X' \to B$, with B Abelian group, we have that there exists a homomorphism if and only if $\sum_{i=1}^{n} k_i f(x_i) = 0$, with $\sum_{i=1}^{n} k_i x_i \in N$.

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Therefore, we can express the group condition using a (possibly infinite) set of equations. We denote it by $E_{A,B,s}$.

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Theorem

Let A_1 *be a finite generated MV-algebra. For any* $x \in \mathrm{A}_1$*, consider the following two real values*

$$p = \sup\{k/m \mid k \in \mathbb{Z}^+, m \in \mathbb{N}, k1_A \leq mx\},$$
$$r = \inf\{l/n \mid l, n \in \mathbb{N}, nx \leq l1_A\}.$$

Then a function $s : \mathrm{A} \to \mathrm{S}$ *is a faithful state if and only if*

1. A_1 *is semisimple*
2. *the set of equations* $E_{s,\mathrm{A},\Gamma^{-1}(S)} \cup \{s(1_A) = 1\}$ *is satisfied*
3. *for all* $x \in \mathrm{A}_1$ *different from* 0_A *and* 1_A *we have* $s(x) \in [p, r] \cap (0, 1)$*.*

The case for $C := \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1,0))$

Definition

Let A be an MV-algebra and $x \in A$. Then x has the downward finite chain property, denoted by $x \in A^{\downarrow}$, if all chains between 0_A and x are finite. We denote by $x \in (A^{\downarrow})_n$ the fact that the number of elements of the longest chain is equal to n.

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Note that x has the downward (upward) chain property if and only if $\neg x$ has the upward (downward) chain property.

Theorem

An equational state of type $(\mathrm{A}, \mathrm{C}, s)$ *is subdirectly irreducible if and only if:*

1. *A has the directional finite chain property;*
2. *the system of equations* $E_{s,\mathrm{A},\mathbb{Z}\times\mathbb{Z}} \cup \{s(1_A) = (1,0)\}$ *is satisfied;*
3. *if* $x \in (A^{\downarrow})_n$ *and* $x \neq 1_A$ *then* $(1,0) > s(x) \geq (0, n-1)$.

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If $x \neq 1_A$ and $x \in (A^{\downarrow})_n$ then, again since s is injective for each chain, we have that $s(x) \geq (0, n-1)$.

The right-to-left implication

We show that $s\colon \Gamma^{-1}(A) \to \mathbb{Z} \times \mathbb{Z}$ is a state such that $s(x) \neq 0$ if $x \neq 0_A$. Note that we know that it is a group homomorphism.

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Let $x \in A$ different from 0 or 1. If it has the downward finite chain then there exists $n \in \mathbb{N}$ such that $x \in (A^{\downarrow})_n$, hence $s(a) \geq (0, n-1)$.

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Otherwise, since x has the upward finite chain property, we derive that since $s(\neg x) < (1, 0)$ then $s(x) > (0, 0)$.

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