Modal Logics of Zariski-constructible and Semialgebraic Sets

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Closure algebras extending algebraic sets

- Algebraic sets:

$$\mathcal{A}(\mathbb{R}^n) = \{X \subseteq \mathbb{R}^n \mid \exists A \subseteq \mathbb{R}[x_1, \ldots, x_n] \text{ such that } X = \mathcal{Z}(A)\}$$

$$\begin{tikzcd} & \mathcal{C}(\mathbb{A}^n) \\ \mathcal{A}(\mathbb{R}^n) \arrow[ur] \arrow[dr] & \\ & \mathcal{SA}(\mathbb{R}^n) \end{tikzcd}$$

$\mathcal{C}(\mathbb{A}^n)$: Zariski-constructible sets

- Boolean envelope of $\mathcal{A}(\mathbb{R}^n)$
- Coincides with the constructibe sets of the affine spaces $\mathbb{A}^n$ with the Zariski topology

$\mathcal{SA}(\mathbb{R}^n)$: Semialgebraic sets

- Closure of $\mathcal{A}(\mathbb{R}^n)$ under projections
- Coincides with the definable sets in $(\mathbb{R}, 0, 1, +, \cdot, <)$

Stratifications

A (finite) stratification of X is a ...

1. ... surjective interior map $\pi : X \to I_\pi$ onto a finite poset with the Alexandrov topology, the *decomposition space*.
2. ... finite partition $X = \bigcup \mathcal{S}$ such that:
   - all $A \in \mathcal{S}$ are locally closed and non-empty,
   - if $A \cap \mathrm{Cl}(B) \neq \emptyset$ then $A \subseteq \mathrm{Cl}(B)$. (frontier condition)

Proposition

There is a one-to-one correspondence between both definitions.

Special classes of stratifications

- Triangulations $\rightsquigarrow$ polyhedral sets
- Nash stratifications $\rightsquigarrow$ semialgebraic sets
  1. the strata A are Nash manifolds, i.e. semialgebraically homeomorphic to $(0,1)^{\dim A}$
  2. $A < B \;\Rightarrow\; \dim A < \dim B$
- Local normal triviality $\rightsquigarrow$???

$\vdots$

Given a class C of stratifications of X, we can define C-objects as sets of the form

$$\bigsqcup \mathcal{A} \text{ for some } \mathcal{A} \subseteq \mathcal{S} \in C.$$

Stratification property

Definition

Let $A \leq \mathcal{C}(X)$ be a modal subalgebra and C be a family of stratifications with strata from A.
We say that A has the *stratification property (with respect to C)* if every finite collection of sets $X_1, \ldots, X_n \in A$ allows a stratification $\mathcal{S}$ (in C) such that all X_i are $\mathcal{S}$-objects.

- For polyhedral sets, this is the 'triangulation lemma'
- If $C = \{\text{all stratifications}\}$, this is equivalent to A locally finite
- $\mathcal{C}(\mathbb{A}^n)$ has the stratification property as a Noetherian space
- $\mathcal{SA}(\mathbb{R}^n)$ has the stratification property w.r.t. $C = \{\text{Nash stratifications}\}$

Stratifications and modal logic

Theorem

Let $A \leq \mathcal{C}(X)$ a modal subalgebra with the stratification property with respect to C. Then:

$$\begin{aligned} \mathsf{Var}(A) &= \mathsf{Var}(\{I_{\mathcal{S}} \mid \mathcal{S} \in C\}) \\ L(A) &= L(\{I_{\mathcal{S}} \mid \mathcal{S} \in C\}) \end{aligned}$$

- Note: $\mathcal{C}(X)$ and A are always Grz algebras

Compact SA sets

$\mathcal{SA}(C)$: Grz algebra of SA subsets of a compact SA set C
$\mathcal{PS}(P)$: Grz algebra of polyhedral sets in a polyhedron P

Theorem (Łojasiewicz[1])

Let $A_1, \ldots, A_n \in \mathcal{SA}(C)$. Then, there exists a polyhedron $P = |\Delta|$ in $\mathbb{R}^n$ with a triangulation Δ and a semialgebraic homeomorphism $h : P \to C$ such that each set A_i is the image of a polyhedral set compatible with Δ.

- Fix some polyhedron $P \cong_{sa} C$
- $\mathcal{PS}(P) \not\models \varphi \Rightarrow \mathcal{SA}(C) \not\models \varphi$
  because polyhedral sets are SA:
  $\varphi(A_1, \ldots, A_k) \neq P \;\Rightarrow\; \varphi(h(A_1), \ldots, h(A_k)) \neq C$

[1] Professor at the Jagiellonian University

Compact SA sets

Conversely, suppose that $\mathcal{SA}(C) \not\models \varphi$. e

Triangulation theorem $\Rightarrow$
There is a polyhedron $Q \cong_{sa} C$ such that $\mathcal{PS}(Q) \not\models \varphi$.

- We know that $P \cong_{sa} Q$

Shiota and Yokoi: 'Triangulations of subanalytic sets and locally subanalytic manifolds' (1984)

Theorem (SA Hauptvermutung)

$P \cong_{sa} Q \;\Rightarrow\; P \cong_{PL} Q$

$\Rightarrow \mathcal{PS}(P) \not\models \varphi$

Compact SA sets

Theorem

Let C be a SA compactum. Then, the SA logic of C equals the polyhedral logic of some polyhedron, which is SA homeomorphic to C and unique up to PL homeomorphism.

$\mathbf{B}^n_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$: closed balls of radius r

- $\mathbf{B}^n_r \cong_{sa} \Delta_n$

$\Rightarrow$ $L(\mathcal{SA}(\mathbf{B}^n_r)) = L(\mathcal{PS}(\Delta_n)) = \mathbf{PL}_n$

Adam-Day et al.: *The Intermediate Logic of Convex Polyhedra* (2023)

- Axiomatization of $\mathbf{PL}_n$

SA logic of the Euclidean spaces

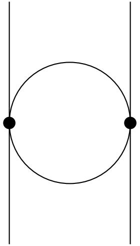
Theorem

$L(\mathcal{SA}(\mathbb{R}^n)) = L(\mathcal{PS}(\Delta_n))$

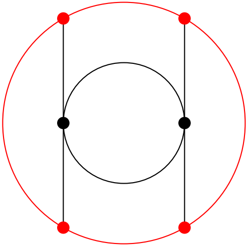
- Sufficient: $L(\mathcal{SA}(\mathbb{R}^n)) = L(\mathcal{SA}(\mathbf{B}^n_r))$ for some r

$\mathcal{SA}(\mathbb{R}^n) \not\models \varphi \Rightarrow \mathcal{SA}(\mathbf{B}^n_r) \not\models \varphi$

Nash stratification $\mathcal{S}$



$\mathcal{S} \cap \mathbf{B}^n_r$ for r large enough



SA logic of the Euclidean spaces

Conversely:

$\mathcal{SA}(\mathbf{B}^n_r) \not\models \varphi \Rightarrow \mathcal{SA}(\mathbb{R}^n) \not\models \varphi$

Define the map $e : \mathbb{R}^n \to \mathbf{B}^n_1, x \mapsto \left\{ \begin{array}{rl} x, & \text{if } \|x\| \leq 1 \\ x/\|x\|^2, & \text{if } \|x\| \geq 1. \end{array} \right.$

- e is interior surjection
- For any stratification $\pi : \mathbf{B}^n_1 \to I$, the composition $e \circ \pi$ is a stratification of $\mathbb{R}^n$ with the same decomposition space

Zariski-constructible sets

Theorem

$L(\mathcal{C}(\mathbb{A}^n)) = \mathbf{Grz.2} \oplus \mathbf{bd}_{n+1}$

<u>Soundness</u>

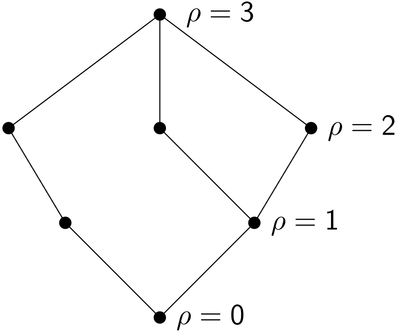
- $\mathcal{C}(\mathbb{A}^n) \in \mathbf{GA}$
  because all decomposition spaces are finite posets
- $\mathcal{C}(\mathbb{A}^n) \models \Diamond\Box p \to \Box\Diamond p$
  because $\mathbb{A}^n$ are irreducible (a.k.a. hyperconnected)
- $\mathcal{C}(\mathbb{A}^n) \models \mathbf{bd}_{n+1}$
  because $\mathbb{A}^n$ have Krull dimension n
  and $A < B \Rightarrow \dim A < \dim B$ for strata A, B

Zariski-constructible sets: Completeness

- **Step 1:** Narrowing down the frames

Proposition

$\mathbf{Grz.2} \oplus \mathbf{bd}_{n+1}$ *is complete w.r.t. finite graded topped trees of height n.*



Zariski-constructible sets: Completeness

- **Step 2:** Existence of weighting functions

Definition

A *weighting function* is a function $w : T - \{t\} \to \mathbb{R}_{\geq 0}$ such that:

If $x \prec y$ and $x \prec y'$ are immediate successors of x, then either $w(z) < w(z')$ for all $z \geq y$ and $z' \geq y'$, or $w(z) > w(z')$ for all $z \geq y$ and $z' \geq y'$.

Lemma

Let T be a finite topped graded tree of height n. Then, there exists a weighting function $w : T - \{t\} \to \mathbb{R}_{\geq 0}$.

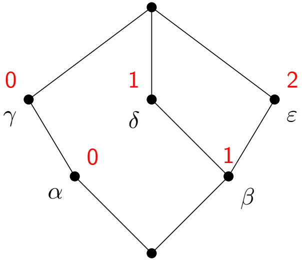
Zariski-constructible sets: Completeness

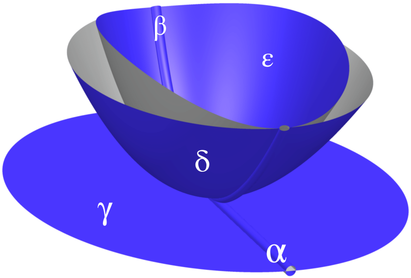
- **Step 3:** Realize a finite graded topped tree T of height n as a poset of algebraic sets in $\mathbb{R}^n$
  - Choose a weighting function w on T
  - Let $a \in T - \{t\}$ with $\downarrow a = \{a_0, \ldots, a_{\rho(a)}\}$
    Assign the polynomial
    $P_a = x_0 - w(a_1)x_1^2 - \ldots - w(a_{\rho(a)})x_{\rho(a)}^2 \in \mathbb{R}[n]$
  - Assign the algebraic set
    $Z_a = \mathcal{Z}\big(\{P_a, x_{\rho(a)+1}, \ldots, x_{n-1}\}\big) \subseteq \mathbb{R}^n$
  - Build the poset
    $Z(T) = \{Z_a \subseteq \mathbb{A}^n \mid a \in T - \{t\}\} \sqcup \{\mathbb{A}^n\}$

Lemma

$T \cong (Z(T), \subseteq)$

Zariski-constructible sets: Completeness





- **Step 4:** Build a stratification of $\mathbb{A}^n$ from $Z(T)$
  Define $S(T) = \{Z - \bigcup \Downarrow Z \mid Z \in Z(T)\}$

Lemma

$S(T)$ is a stratification and $T \cong I_{S(T)}$

Modal logics of $\mathcal{C}(\mathbb{A}^n)$ and $\mathcal{SA}(\mathbb{R}^n)$

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{c} L(\mathcal{C}(\mathbb{A})) \\ = \mathbf{Grz.2} \oplus \mathbf{bd}_2 \end{array} \arrow[dl, dash] \arrow[dr, dash] & & & \\
\begin{array}{c} L(\mathcal{SA}(\mathbb{R})) \\ = \mathbf{PL}_1 \end{array} \arrow[dr, dash] & & \begin{array}{c} L(\mathcal{C}(\mathbb{A}^2)) \\ = \mathbf{Grz.2} \oplus \mathbf{bd}_3 \end{array} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \begin{array}{c} L(\mathcal{SA}(\mathbb{R}^2)) \\ = \mathbf{PL}_2 \end{array} \arrow[dr, dash] & & \begin{array}{c} L(\mathcal{C}(\mathbb{A}^3)) \\ = \mathbf{Grz.2} \oplus \mathbf{bd}_4 \end{array} \arrow[dl, dash] \arrow[dr, dash, dashed] & \\
 & & \begin{array}{c} L(\mathcal{SA}(\mathbb{R}^3)) \\ = \mathbf{PL}_3 \end{array} \arrow[dr, dash, dashed] & & {} \\
 & & & {} &
\end{tikzcd}$$