Extending Rewriting to Arbitrary Monoids

Eduardo Magalhães

CMUP, Departamento de Matemática,
Faculdade de Ciências, Universidade do Porto

CENTRO DE
MATEMÁTICA
UNIVERSIDADE DO PORTO

String Rewriting Systems I

Definition

A *string rewriting system* (or historically *semi-Thue system*) is a pair (A, R) where A is a set called the *alphabet* and $R \subseteq F_A \times F_A$ is a binary relation, where F_A denotes the free monoid with generators A.

Note: We will use the notation (F_A, R) for a string rewriting system (A, R).

Given a rewriting system (F_A, R) and words $u, v \in F_A$, we write $u \to_R v$ if there are $x, y \in F_A$ and $(r, s) \in R$ such that

$$u = xry, \quad v = xsy.$$

We write $u \to_R^* v$ if there are $v_0, \ldots, v_m \in F_A$, for some $m \geq 0$, such that

$$u = v_0 \to_R \cdots \to_R v_m = v.$$

String Rewriting Systems II

Definition

Given a rewriting system (F_A, R), we say that it is:

- *Confluent*, if given any $a, b, c \in F_A$ such that $a \to_R^* b$ and $a \to_R^* c$, then there exists some $d \in F_A$ such that $b \to_R^* d$ and $c \to_R^* d$;
- *Noetherian* if there is no infinite chain $w_1 \to_R w_2 \to_R \ldots$.

Furthermore, we say that an element $a \in F_A$ is *irreducible* if there is no $v \in F_A$ such that $a \to_R v$.

String Rewriting Systems III

Remark

Given a Noetherian confluent rewriting system (F_A, R), then for any $a \in F_A$, there exists a unique irreducible $\bar{a} \in F_A$ such that $a \rightarrow_R^* \bar{a}$. We call $\bar{a}$ the normal form of a.
Furthermore, we denote the set of all irreducible words of F_A by $\overline{F_A}$.

Example:

• Let $A = \{a, b\}$ and $R = \{(ba, ab)\}$. This is confluent and Noetherian. Irreducible words are those of the form $a^n b^m$, for $n, m \geq 0$.

$$baba \rightarrow_R abba \rightarrow_R abab \rightarrow_R aabb$$

String Rewriting Systems IV

Definition

Let (F_A, R) be a string rewriting system. We define the binary relation $\leftrightarrow_R^*$ on F_A, to be the symmetric reflexive transitive closure of $\rightarrow_R$. The relation $\leftrightarrow_R^*$ is called the *Thue congruence*.

Definition

Given a string rewriting system (F_A, R), we say that a monoid M is *presented* by (F_A, R) if $M \cong F_A / \leftrightarrow_R^*$.

Proposition

Consider a Noetherian confluent rewriting system (F_A, R). Define a binary operation $\bar{\cdot}$ on the set of irreducibles $\overline{F_A}$ as $a \,\bar{\cdot}\, b = \overline{a \cdot b}$. Then

$$(\overline{F_A}, \bar{\cdot}\,) \cong F_A / \leftrightarrow_R^*$$

Motivation

Question: Can we study the first-order properties of a monoid/group via the first-order properties of its presentation?

Motivation

Question: Can we study the first-order properties of a monoid/group via the first-order properties of its presentation?

Question

Consider a monoid $(M, \cdot)$ viewed as a first-order structure over the language $\mathcal{L} = \{\cdot\}$ and consider a string rewriting system $(F_A, \oplus, R)$ that presents it, viewed as a first-order structure over the language $\mathcal{K} = \{\cdot, R\}$.
What can we assert about the first-order behavior of $(M, \cdot)$ from the study of the first-order behavior of $(F_A, \oplus, R)$?

Motivation

Question: Can we study the first-order properties of a monoid/group via the first-order properties of its presentation?

Question

Consider a monoid $(M, \cdot)$ viewed as a first-order structure over the language $\mathcal{L} = \{\cdot\}$ and consider a string rewriting system $(F_A, \oplus, R)$ that presents it, viewed as a first-order structure over the language $\mathcal{K} = \{\cdot, R\}$.
What can we assert about the first-order behavior of $(M, \cdot)$ from the study of the first-order behavior of $(F_A, \oplus, R)$?

Remark

The class of free-monoids is not axiomatizable in first-order logic.

Monoidal Rewriting Systems

Definition

A *monoidal rewriting system (MRS)* is a triple $(M, \cdot, R)$ where $(M, \cdot)$ is a monoid and $R \subseteq M \times M$ is a binary relation.

Given a MRS $(M, \cdot, R)$ and elements $u, v \in M$, we write $u \rightarrow_R v$ if there are $x, y \in M$ and $(r, s) \in R$ such that

$$u = xry, \quad v = xsy.$$

We write $u \rightarrow_R^* v$ if there are $v_0, \ldots, v_m \in M$, for some $m \geq 0$, such that

$$u = v_0 \rightarrow_R \cdots \rightarrow_R v_m = v.$$

The Thue Congruence

Definition

Given a MRS $(M, \cdot, R)$, we define the binary relation $\leftrightarrow_R^*$ to be the symmetric reflexive transitive closure of $\rightarrow_R$.

Proposition

For any MRS $(M, \cdot, R)$, the equivalence relation $\leftrightarrow_R^*$ is a congruence on the monoid M.

Definition

Given a MRS $(M, \cdot, R)$, we say that a monoid N is *presented* by $(M, \cdot, R)$ if $N \cong M / \leftrightarrow_R^*$.

Normal Forms

Definition

Given a MRS $(M, \cdot, R)$, we say that it is:

- *Confluent*, if given any $a, b, c \in M$ such that $a \to_R^* b$ and $a \to_R^* c$, there exists some $d \in M$ such that $b \to_R^* d$ and $c \to_R^* d$;
- *Noetherian* if any infinite chain $w_1 \to_R w_2 \to_R \ldots$ eventually stabilizes.

Furthermore, we say that an element $a \in M$ is *irreducible* if for any $v \in M$, $a \to_R v$ implies that $a = v$.

Proposition

Given a Noetherian Confluent MRS $(M, \cdot, R)$, then for any $a \in M$, there exists a unique irreducible $\bar{a} \in M$ such that $a \to_R^* \bar{a}$. We call $\bar{a}$ the normal form of a. We denote the set of all irreducible elements of M by $\overline{M}$.

The I operator

Lemma

Let $(M, \cdot, R)$ be a Noetherian confluent MRS. Consider the binary operation on $\overline{M}$ given by:

$$\begin{aligned} \bar{\cdot} : \overline{M} \times \overline{M} &\to \overline{M} \\ (u, v) &\mapsto \overline{u \cdot v} \end{aligned}$$

Then $(\overline{M}, \bar{\cdot}\,)$ is a monoid, which we call the monoid of irreducibles of $(M, \cdot, R)$ and denote by $I(M, \cdot, R)$.

Proposition

Let $(M, \cdot, R)$ be a Noetherian confluent MRS. Then $(M, \cdot, R)$ presents $I(M, \cdot, R)$, i.e.

$$I(M, \cdot, R) \cong M/ \leftrightarrow_R^*$$

Examples

- Let $M = \mathbb{N}_{>0} \times \mathbb{N}_{>0}$, with $(a, b) \cdot (c, d) = (ac, bd)$ and

$$R = \{((p, p), (1, 1)) : p \text{ prime}\}$$

Then $I(M, \cdot, R) \;\cong\; (\mathbb{Q}_{>0}, \times)$

Examples

- Let $M = \mathbb{N}_{>0} \times \mathbb{N}_{>0}$, with $(a, b) \cdot (c, d) = (ac, bd)$ and

$$R = \{((p, p), (1, 1)) : p \text{ prime}\}$$

Then $I(M, \cdot, R) \;\cong\; (\mathbb{Q}_{>0}, \times)$

- Fix a topological space (X, τ) and consider the monoid $(\mathcal{P}X, \cup)$. Let

$$R = \{(U, \mathrm{cl}(U)) : U \subseteq X\}$$

Then, in $\overline{\mathcal{P}X} = \{\text{Closed subsets of } X\}$.
So $(\mathcal{P}X, \cup, R)$ is a presentation for the monoid of closed subsets of X under union.

MRS-homomorphisms

Definition

Let $\mathcal{M} = (M, \cdot, R)$ and $\mathcal{N} = (N, *, L)$ be two MRS. A *MRS-homomorphism* $\phi : \mathcal{M} \to \mathcal{N}$ is a monoid homomorphism $\phi : (M, \cdot) \to (N, *)$ such that, for any $u, v \in M$, we have:

$$(u, v) \in R \implies \phi(u) \to_L^* \phi(v)$$

MRS-homomorphisms

Definition

Let $\mathcal{M} = (M, \cdot, R)$ and $\mathcal{N} = (N, *, L)$ be two MRS. A *MRS-homomorphism* $\phi : \mathcal{M} \to \mathcal{N}$ is a monoid homomorphism $\phi : (M, \cdot) \to (N, *)$ such that, for any $u, v \in M$, we have:

$$(u, v) \in R \implies \phi(u) \to_L^* \phi(v)$$

MRS $\to$ *category of MRS and MRS-homomorphisms*
NCRS $\to$ *full subcategory consisting of all Noetherian confluent MRS.*

The functor I

Lemma

Let $(M, \cdot, R)$ and $(N, *, L)$ be Noetherian confluent MRS, and $\phi : (M, \cdot, R) \to (N, *, L)$ a MRS-homomorphism. Then, the map

$$\begin{aligned} I\phi : (\overline{M}, \bar{\cdot}) &\to (\overline{N}, \bar{*}) \\ u &\mapsto \overline{\phi(u)} \end{aligned}$$

is a monoid homomorphism. Furthermore, I is functorial.

The functor I

Lemma

Let $(M, \cdot, R)$ and $(N, *, L)$ be Noetherian confluent MRS, and $\phi : (M, \cdot, R) \to (N, *, L)$ a MRS-homomorphism. Then, the map

$$\begin{aligned} I\phi : (\overline{M}, \bar{\cdot}) &\to (\overline{N}, \bar{*}) \\ u &\mapsto \overline{\phi(u)} \end{aligned}$$

is a monoid homomorphism. Furthermore, I is functorial.

So, we have a functor

$$I : \mathbf{NCRS} \to \mathbf{Mon}$$

where **Mon** is the category of monoids.

The G operator

Let $(M, \cdot)$ be a monoid and let M^+ denote the set $M \setminus \{1\}$.
Let $\nu_M : M \hookrightarrow F_{M^+}$ be the injective function defined by $\nu_M(1) = \varepsilon$ and $\nu_M(m) = m$.

Definition

Given a monoid $(M, \cdot)$, we define $G(M)$ as the MRS $(F_{M^+}, \oplus, R_M)$, where $\oplus$ denotes concatenation and

$$R_M = \{(\nu_M(a) \oplus \nu_M(b), \nu_M(a \cdot b)) : a, b \in M\}$$

Lemma

For any monoid $(M, \cdot)$, the MRS $G(M)$ is Noetherian and confluent.

The G functor

Given two monoids $(M, \cdot)$ and $(N, *)$ and a homomorphism $\phi : (M, \cdot) \to (N, *)$, we can define a monoid homomorphism $G(\phi) : F_{M^+} \to F_{N^+}$:

$$\begin{tikzcd}
M^+ \arrow[r, "\phi|_{M^+}"] \arrow[d, hook, "\iota"'] & N \arrow[d, hook, "\nu_N"] \\
F_{M^+} \arrow[r, dashed, "G\phi"'] & F_{N^+}
\end{tikzcd}$$

Lemma

Given a monoid homomorphism $\phi : (M, \cdot) \to (N, *)$, then $G(\phi) : G(M) \to G(N)$ is a MRS-homomorphism. Furthermore, G is functorial.

The G functor

Given two monoids $(M, \cdot)$ and $(N, *)$ and a homomorphism $\phi : (M, \cdot) \to (N, *)$, we can define a monoid homomorphism $G(\phi) : F_{M^+} \to F_{N^+}$:

$$\begin{tikzcd}
M^+ \arrow[r, "\phi|_{M^+}"] \arrow[d, hook, "\iota"'] & N \arrow[d, hook, "\nu_N"] \\
F_{M^+} \arrow[r, dashed, "G\phi"'] & F_{N^+}
\end{tikzcd}$$

Lemma

Given a monoid homomorphism $\phi : (M, \cdot) \to (N, *)$, then $G(\phi) : G(M) \to G(N)$ is a MRS-homomorphism. Furthermore, G is functorial.

As such, we have a functor

$$G : \mathbf{Mon} \to \mathbf{NCRS}$$

What is the relationship between G and I?

Consider $A = (\mathbb{N}, +, R)$ where $R = \{(2, 0)\}$.
Then

$$\hom_{\mathbf{NCRS}}(G(\mathbb{Z}_2), A) \ncong \hom_{\mathbf{Mon}}(\mathbb{Z}_2, I(A))$$

What is the relationship between G and I?

Consider $A = (\mathbb{N}, +, R)$ where $R = \{(2,0)\}$.
Then

$$\hom_{\mathbf{NCRS}}(G(\mathbb{Z}_2), A) \ncong \hom_{\mathbf{Mon}}(\mathbb{Z}_2, I(A))$$

Indeed:

- $I(A) \cong \mathbb{Z}_2$, and so

$$|\hom_{\mathbf{Mon}}(\mathbb{Z}_2, I(A))| = 2$$

What is the relationship between G and I?

Consider $A = (\mathbb{N}, +, R)$ where $R = \{(2,0)\}$.
Then

$$\hom_{\mathbf{NCRS}}(G(\mathbb{Z}_2), A) \ncong \hom_{\mathbf{Mon}}(\mathbb{Z}_2, I(A))$$

Indeed:

- $I(A) \cong \mathbb{Z}_2$, and so

$$|\hom_{\mathbf{Mon}}(\mathbb{Z}_2, I(A))| = 2$$

- $G(\mathbb{Z}_2) = (F_{\{1\}}, \oplus, R_M)$, where $R_M = \{(11, \varepsilon), (1,1)\}$, and so

$$|\hom_{\mathbf{NCRS}}(G(\mathbb{Z}_2), A)| = \aleph_0$$

$\mathbf{NCRS}_2$

Definition

We give $\mathbf{NCRS}_2$ a (strict) 2-categorical structure in the following way:

- The 0-cells are Noetherian confluent MRS;
- The 1-cells are MRS-homomorphisms;
- Given two MRS-homomorphisms $f, g : (A, \cdot, R) \to (B, *, L)$, there exists a unique 2-cell $\tau : f \Rightarrow g$ if and only if, for all $a \in A$, we have $f(a) \leftrightarrow_L^* g(a)$

$\mathbf{NCRS}_2$

Definition

We give $\mathbf{NCRS}_2$ a (strict) 2-categorical structure in the following way:

- The 0-cells are Noetherian confluent MRS;
- The 1-cells are MRS-homomorphisms;
- Given two MRS-homomorphisms $f, g : (A, \cdot, R) \to (B, *, L)$, there exists a unique 2-cell $\tau : f \Rightarrow g$ if and only if, for all $a \in A$, we have $f(a) \leftrightarrow_L^* g(a)$

Remark

There is at most one 2-cell between any two morphisms, and all 2-cells are invertible.

$\mathbf{NCRS}_2$

Definition

We give $\mathbf{NCRS}_2$ a (strict) 2-categorical structure in the following way:

- The 0-cells are Noetherian confluent MRS;
- The 1-cells are MRS-homomorphisms;
- Given two MRS-homomorphisms $f, g : (A, \cdot, R) \to (B, *, L)$, there exists a unique 2-cell $\tau : f \Rightarrow g$ if and only if, for all $a \in A$, we have $f(a) \leftrightarrow_L^* g(a)$

Remark

There is at most one 2-cell between any two morphisms, and all 2-cells are invertible.

We also consider **Mon** as a strict 2-category by considering the hom-categories $\hom_{\mathbf{Mon}}(M, N)$ as being discrete.

2-functors I and G

Lemma

Let $\phi, \psi : (M, \cdot, R) \to (N, *, L)$ be two MRS-homomorphisms such that $\phi(x) \leftrightarrow_L^* \psi(x)$ for all $x \in M$. Then $I\phi = I\psi$.

In other words, if there exists a 2-cell $\tau : \phi \Rightarrow \psi$, then $I\phi = I\psi$.

2-functors I and G

Lemma

Let $\phi, \psi : (M, \cdot, R) \to (N, *, L)$ be two MRS-homomorphisms such that $\phi(x) \leftrightarrow_L^* \psi(x)$ for all $x \in M$. Then $I\phi = I\psi$.

In other words, if there exists a 2-cell $\tau : \phi \Rightarrow \psi$, then $I\phi = I\psi$.

As such, we can extend I to a strict 2-functor $I : \mathbf{NCRS}_2 \to \mathbf{Mon}$ in the obvious way. Additionally, we can also trivially extend G to a strict 2-functor $G : \mathbf{Mon} \to \mathbf{NCRS}_2$.

Biadjunction

Theorem

The pair (G, I) forms a biadjunction, where G is left biadjoint and I is right biadjoint.

$$\begin{tikzcd} \mathbf{Mon} \arrow[r, bend left, "G"] \arrow[r, phantom, "\perp"] & \mathbf{NCRS_2} \arrow[l, bend left, "I"] \end{tikzcd}$$

Concretely, the unit $\eta : 1_{\mathbf{Mon}} \to IG$ is the identity 2-natural transformation (in particular $IG = 1_{\mathbf{Mon}}$), and the counit $\varepsilon : GI \to 1_{\mathbf{NCRS_2}}$ is a strong transformation. Furthermore, they satisfy the triangle conditions strictly.

Generalized elementary Tietze transformations (GETT)

Question

Given a fixed monoid M, what can be said about the MRS's $(A, \cdot, R)$ such that $I(A, \cdot, R) \cong M$?

Generalized elementary Tietze transformations (GETT)

Question

Given a fixed monoid M, what can be said about the MRS's $(A, \cdot, R)$ such that $I(A, \cdot, R) \cong M$?

Definition

Let $(A, \cdot, R)$ be a MRS. The following 'procedures' are called *generalized elementary Tietze transformations* (GETTs):

1. Given $a, b \in A$ with $a \leftrightarrow_R^* b$, then $(A, \cdot, R \cup \{(a, b)\})$ is obtained from $(A, \cdot, R)$ by a GETT of type 1;

Generalized elementary Tietze transformations (GETT)

Question

Given a fixed monoid M, what can be said about the MRS's $(A, \cdot, R)$ such that $I(A, \cdot, R) \cong M$?

Definition

Let $(A, \cdot, R)$ be a MRS. The following 'procedures' are called *generalized elementary Tietze transformations* (GETTs):

1. Given $a, b \in A$ with $a \leftrightarrow_R^* b$, then $(A, \cdot, R \cup \{(a, b)\})$ is obtained from $(A, \cdot, R)$ by a GETT of type 1;
2. Given $(a, b) \in R$ with $a \leftrightarrow^*_{R \setminus \{(a,b)\}} b$, then $(A, \cdot, R \setminus \{(a, b)\})$ is obtained from $(A, \cdot, R)$ by a GETT of type 2;

Generalized elementary Tietze transformations (GETT)

Question

Given a fixed monoid M, what can be said about the MRS's $(A, \cdot, R)$ such that $I(A, \cdot, R) \cong M$?

Definition

Let $(A, \cdot, R)$ be a MRS. The following 'procedures' are called *generalized elementary Tietze transformations* (GETTs):

1. Given $a, b \in A$ with $a \leftrightarrow^*_R b$, then $(A, \cdot, R \cup \{(a, b)\})$ is obtained from $(A, \cdot, R)$ by a GETT of type 1;
2. Given $(a, b) \in R$ with $a \leftrightarrow^*_{R \setminus \{(a,b)\}} b$, then $(A, \cdot, R \setminus \{(a, b)\})$ is obtained from $(A, \cdot, R)$ by a GETT of type 2;
3. Let $a \in A$ and v be a new symbol, i.e. $v \notin A$. Then $(A * F_v, \cdot, R \cup \{(v, a)\})$ is obtained from $(A, \cdot, R)$ by a GETT of type 3;

Type 4 GETT

Definition

Let $(M, \cdot, R)$ and $J \subseteq R$. A MRS is said to be obtained from $(M, \cdot, R)$ by a type 4 GETT if it is the result of applying only the rules present in J, while keeping the rest of the rules.

Type 4 GETT

Definition

Let $(M, \cdot, R)$ and $J \subseteq R$. A MRS is said to be obtained from $(M, \cdot, R)$ by a type 4 GETT if it is the result of applying only the rules present in J, while keeping the rest of the rules.

Proposition

Performing a GETT of types 1,2,3 or 4 on a MRS does not change the monoid being presented.

Classification of the fiber $I^{-1}(M)$

Theorem

Let $(M, \cdot)$ be a fixed monoid and let $(A, \cdot, R)$ and $(B, *, L)$ be two infinite Noetherian confluent MRS such that $I(A, \cdot, R) = (M, \cdot) = I(B, *, L)$. Furthermore, assume that $|B| = \lambda$. Then $(B, *, L)$ can be obtained from $(A, \cdot, R)$ by performing at most λ generalized elementary Tietze transformations.

Proof sketch:

Classification of the fiber $I^{-1}(M)$

Theorem

Let $(M, \cdot)$ be a fixed monoid and let $(A, \cdot, R)$ and $(B, *, L)$ be two infinite Noetherian confluent MRS such that $I(A, \cdot, R) = (M, \cdot) = I(B, *, L)$. Furthermore, assume that $|B| = \lambda$. Then $(B, *, L)$ can be obtained from $(A, \cdot, R)$ by performing at most λ generalized elementary Tietze transformations.

Proof sketch:

$$\begin{tikzcd}
 & (F_{B^+}, \oplus, W \cup R_M) \arrow[r] & (F_{B^+}, \oplus, L \cup R_M) \arrow[r] & (F_{B^+}, \oplus, L \cup R_B) \arrow[d] \\
(A, \cdot, R) \arrow[d] & & & (B, *, L) \\
(M, \cdot, \emptyset) \arrow[uur, bend right] & & &
\end{tikzcd}$$

Conclusion

Question

Given a fixed monoid M, what can be said about the MRS's $(A, \cdot, R)$ such that $I(A, \cdot, R) = M$?

Answer:

Conclusion

Question

Given a fixed monoid M, what can be said about the MRS's $(A, \cdot, R)$ such that $I(A, \cdot, R) = M$?

Answer:

Corollary

Let $(A, \cdot, R)$ and $(B, *, L)$ be two Noetherian confluent MRS. Then $I(A, \cdot, R) \simeq I(B, *, L)$ if and only if one can be obtained from the other via a (possibly infinite) sequence of GETTs.

Future work: Model Theory of MRS

Question

Consider a monoid N viewed as a first-order structure over the language $\mathcal{L} = \{\cdot\}$ and consider a MRS $(M, \cdot, R)$ that presents it, viewed as a first-order structure over the language $\mathcal{K} = \{\cdot, R\}$.

How much can we know about the first-order behavior of N from the first-order behavior of $(M, \cdot, R)$?

Future work: Model Theory of MRS

Question

Consider a monoid N viewed as a first-order structure over the language $\mathcal{L} = \{\cdot\}$ and consider a MRS $(M, \cdot, R)$ that presents it, viewed as a first-order structure over the language $\mathcal{K} = \{\cdot, R\}$.

How much can we know about the first-order behavior of N from the first-order behavior of $(M, \cdot, R)$?

Proposition

Let $(A, \cdot, R) \preceq (B, *, L)$ be two Noetherian confluent MRS. Then $I(A, \cdot, R) \preceq_{\Sigma_1} I(B, *, L)$ and $I(A, \cdot, R) \preceq_{\Pi_1} I(B, *, L)$.

Future work: Model Theory of MRS

Definition

Let $\mathcal{M}$ be a Noetherian confluent MRS. We let $\mathcal{M}_{\mathrm{nf}}$ denote the structure obtained from $\mathcal{M}$ by adding a new unary functional symbol nf, which is interpreted in $\mathcal{M}$ as the function that sends each element to its normal form.

Future work: Model Theory of MRS

Definition

Let $\mathcal{M}$ be a Noetherian confluent MRS. We let $\mathcal{M}_{\mathrm{nf}}$ denote the structure obtained from $\mathcal{M}$ by adding a new unary functional symbol nf, which is interpreted in $\mathcal{M}$ as the function that sends each element to its normal form.

Proposition

Let $\mathcal{M} \preceq \mathcal{N}$ be two Noetherian confluent MRS. Then $\mathcal{M}_{\mathrm{nf}} \preceq_{\Sigma_1} \mathcal{N}_{\mathrm{nf}}$ and $\mathcal{M}_{\mathrm{nf}} \preceq_{\Pi_1} \mathcal{N}_{\mathrm{nf}}$.

Future work: Model Theory of MRS

Definition

Let $\mathcal{M}$ be a Noetherian confluent MRS. We let $\mathcal{M}_{\mathrm{nf}}$ denote the structure obtained from $\mathcal{M}$ by adding a new unary functional symbol nf, which is interpreted in $\mathcal{M}$ as the function that sends each element to its normal form.

Proposition

Let $\mathcal{M} \preceq \mathcal{N}$ be two Noetherian confluent MRS. Then $\mathcal{M}_{\mathrm{nf}} \preceq_{\Sigma_1} \mathcal{N}_{\mathrm{nf}}$ and $\mathcal{M}_{\mathrm{nf}} \preceq_{\Pi_1} \mathcal{N}_{\mathrm{nf}}$.

$$
\begin{tikzcd}
 & \mathcal{M} \preceq \mathcal{N} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
I\mathcal{M} \preceq_{\Sigma_1} I\mathcal{N} \arrow[rr, leftrightarrow, "?"] & & \mathcal{M}_{\mathrm{nf}} \preceq_{\Sigma_1} \mathcal{N}_{\mathrm{nf}}
\end{tikzcd}
$$

Future work: Model Theory of MRS

Proposition

Let $\mathcal{M}$ be a Noetherian confluent MRS and $\phi(\bar{v})$ a formula in the pure language of monoids. Then there exists a formula $\phi_{\mathrm{nf}}(\bar{v})$ using the unary functional symbol nf such that, for any tuple of irreducible elements $\bar{a}$ of $\mathcal{M}$, we have that $I\mathcal{M} \models \phi[\bar{a}]$ if and only if $\mathcal{M}_{\mathrm{nf}} \models \phi_{\mathrm{nf}}[\bar{a}]$.

Future work: Model Theory of MRS

Proposition

Let $\mathcal{M}$ be a Noetherian confluent MRS and $\phi(\bar{v})$ a formula in the pure language of monoids. Then there exists a formula $\phi_{\mathrm{nf}}(\bar{v})$ using the unary functional symbol nf such that, for any tuple of irreducible elements $\bar{a}$ of $\mathcal{M}$, we have that $I\mathcal{M} \models \phi[\bar{a}]$ if and only if $\mathcal{M}_{\mathrm{nf}} \models \phi_{\mathrm{nf}}[\bar{a}]$.

Corollary

Let $\mathcal{M}$ and $\mathcal{N}$ be two Noetherian confluent MRS.

1. *If $\mathcal{M}_{\mathrm{nf}} \equiv \mathcal{N}_{\mathrm{nf}}$, then $I\mathcal{M} \equiv I\mathcal{N}$;*
2. *If $\mathcal{M}_{\mathrm{nf}} \preceq \mathcal{N}_{\mathrm{nf}}$, then $I\mathcal{M} \preceq I\mathcal{N}$;*
3. *If $f : \mathcal{M}_{\mathrm{nf}} \to \mathcal{N}_{\mathrm{nf}}$ is an elementary embedding, then $I(f) : I\mathcal{M} \to I\mathcal{N}$ is also elementary.*

Acknowledgements

The author was partially supported by CMUP, member of LASI, which is financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the project with reference UID/00144/2025 and associated DOI given by https://doi.org/10.54499/UID/00144/2025.

Strong Transformations

Let $\mathbf{C}, \mathbf{D}$ be two 2-categories and $F, G : \mathbf{C} \to \mathbf{D}$ two strict two functors.
A *strong transformation* $\alpha : F \Rightarrow G$ consists of:

- For each object $X \in \mathbf{C}$, a 1-cell $\alpha_X : F(X) \to G(X)$;
- For each arrow $f : X \to Y$ in $\mathbf{C}$, a 2-cell: $\alpha_f : Gf \circ \alpha_X \Rightarrow \alpha_Y \circ Ff$

Such that, for every 2-cell $\theta : f \to g$ in $\mathbf{C}(X, Y)$, the following equality of pasting diagrams holds:

$$
\begin{tikzcd}
FX \arrow[rr, "\alpha_X"] \arrow[d, bend right=50, "Fg"'] \arrow[d, bend left=50, "Ff"] \arrow[d, phantom, "\overset{F\theta}{\Leftarrow}"] \arrow[drr, phantom, "\swarrow \alpha_f"] & & GX \arrow[d, bend left=50, "Gf"] \\
FY \arrow[rr, "\alpha_Y"'] & & GY
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FX \arrow[rr, "\alpha_X"] \arrow[d, bend right=50, "Fg"'] \arrow[drr, phantom, "\alpha_g \swarrow"] & & GX \arrow[d, bend right=50, "Gg"'] \arrow[d, bend left=50, "Gf"] \arrow[d, phantom, "\overset{G\theta}{\Leftarrow}"] \\
FY \arrow[rr, "\alpha_Y"'] & & GY
\end{tikzcd}
$$

Biadjunction

Let $\mathbf{C}$ and $\mathbf{D}$ be strict 2-categories and consider two strict 2-functors $L : \mathbf{C} \to \mathbf{D}$ (left adjoint) and $R : \mathbf{D} \to \mathbf{C}$ (right adjoint);

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "R"] \end{tikzcd}$$

A *biadjunction* is given by two strong transformations $\eta : 1_{\mathbf{C}} \to RL$ (unit) and $\varepsilon : LR \to 1_{\mathbf{D}}$ (counit) such that η and ε satisfy the triangle identities up to invertible modifications, known as triangulator modifications. Furthermore, we require these triangulator modifications to be coherent, i.e. we require them to satisfy the swallowtail identities.

The counit $\varepsilon : GI \to 1_{\mathbf{NCRS_2}}$

The *counit*

$$\varepsilon : GI \to 1_{\mathbf{NCRS_2}}$$

is defined as follows:

- Given an object $(A, \cdot, R)$, we define ε_A to be the 1-cell:

$$\begin{aligned} \varepsilon_A : GI(A, \cdot, R) &\to (A, \cdot, R) \\ a_1 \oplus \ldots \oplus a_n &\mapsto a_1 \cdot \ldots \cdot a_n \end{aligned}$$

- Given a 1-cell $\phi : (A, \cdot, R) \to (B, *, L)$, we define ε_ϕ to be the unique 2-cell:

$$\begin{tikzcd} GI(A, \cdot, R) \arrow[r, "\varepsilon_A"] \arrow[d, "GI\phi"'] \arrow[dr, phantom, "\swarrow"] & (A, \cdot, R) \arrow[d, "\phi"] \\ GI(B, *, L) \arrow[r, "\varepsilon_B"'] & (B, *, L) \end{tikzcd}$$

Example 1

For $n \geq 1$, let $C_n = \{c_0^n, c_1^n, \ldots, c_n^n, t_n\}$. Let $S^* = \bigcup_{n \geq 1} C_n$ with multiplication defied as $x \cdot y = x$. Let M be the monoid obtained from S^* by adding an identity.
Let $\mathcal{M} = (M, \cdot, R)$ where

$$R = \{((c_{i+1}^n, c_i^n)) : n \geq 1 \text{ and } 1 \leq i < n\} \cup \{(t_n, c_n^n), (t_n, c_{n-1}^n) : n \geq 1\}$$

Define $\mathcal{N} = (N, \cdot, L)$ from $\mathcal{M}$ by adding $D = \{d_0, d_1, \ldots\}$ as elements and

$$L = R \cup \{(d_{n+1}, d_n) : n \geq 1\}$$

$$
\begin{tikzcd}
 & & & t_3 \arrow[d] \arrow[dd, bend right] & & d_4 \arrow[d] \\
 & & t_2 \arrow[d] \arrow[dd, bend right] & c_3^3 \arrow[d] & & d_3 \arrow[d] \\
 & t_1 \arrow[d] \arrow[dd, bend right] & c_2^2 \arrow[d] & c_2^3 \arrow[d] & \cdots & d_2 \arrow[d] \\
 & c_1^1 \arrow[d] & c_1^2 \arrow[d] & c_1^3 \arrow[d] & & d_1 \arrow[d] \\
1 & c_0^1 & c_0^2 & c_0^3 & & d_0
\end{tikzcd}
$$

$$\forall x \exists yzw\, ((x = 1) \vee (\mathrm{nf}(x) = \mathrm{nf}(y) \wedge R(y, z) \wedge R(y, w) \wedge z \neq w))$$

Future work: Upper Bounded MRS

Definition

Let M be a MRS. We say that M is *upper bounded* if there exists some N such that every rewriting chain of sequentially distinct elements has length less then N.

Future work: Upper Bounded MRS

Definition

Let M be a MRS. We say that M is *upper bounded* if there exists some N such that every rewriting chain of sequentially distinct elements has length less then N.

Proposition

Let M be a MRS. Then M admits a non noetherian elementary extension if and only if M is not upper bounded.

Future work: Upper Bounded MRS

Definition

Let M be a MRS. We say that M is *upper bounded* if there exists some N such that every rewriting chain of sequentially distinct elements has length less then N.

Proposition

Let M be a MRS. Then M admits a non noetherian elementary extension if and only if M is not upper bounded.

Proposition

Let $\mathcal{M} = (M, \cdot, R)$ and $\mathcal{N} = (N, \cdot, L)$ be two Noetherian confluent MRS, where $\mathcal{M}$ is upper bounded. If $\mathcal{M} \preceq \mathcal{N}$ then $\mathcal{M}_{\mathrm{nf}} \preceq \mathcal{N}_{\mathrm{nf}}$.

Future work: Upper Bounded MRS

Definition

Let M be a MRS. We say that M is *upper bounded* if there exists some N such that every rewriting chain of sequentially distinct elements has length less then N.

Proposition

Let M be a MRS. Then M admits a non noetherian elementary extension if and only if M is not upper bounded.

Proposition

Let $\mathcal{M} = (M, \cdot, R)$ and $\mathcal{N} = (N, \cdot, L)$ be two Noetherian confluent MRS, where $\mathcal{M}$ is upper bounded. If $\mathcal{M} \preceq \mathcal{N}$ then $\mathcal{M}_{\mathrm{nf}} \preceq \mathcal{N}_{\mathrm{nf}}$.

Remark

Every string rewriting system $(F_A, \oplus, R)$ that is upper bounded is trivial, i.e. if R contains a rule of the form (u, v) with $u \neq v$, then (F_A, R) is not upper bounded.