Clonoids over finite vector spaces and related computational problems

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Introduction

Let A be a set and $\mathcal{A} \subseteq \mathcal{O}_A = \bigcup A^{A^n}$. $\mathcal{A}$ is a clone on A if it contains all projections and is closed under composition, i.e.

- $\pi_i^n \in \mathcal{A}$, for every $1 \leq i \leq n$ and $n \in \mathbb{N}$, where $\pi_i^n(x_1, \ldots, x_n) = x_i$
- $f \in \mathcal{A}^{(n)}, g_1, \ldots, g_n \in \mathcal{A}^{(k)} \Rightarrow f \circ (g_1, \ldots, g_n) \in \mathcal{A}^{(k)}$ for all $k, n \in \mathbb{N}$.

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Let $\mathcal{A}$, $\mathcal{B}$ be a clones on sets A, B. We then say that $\mathcal{C} \subseteq \mathcal{O}_{A,B} = \bigcup B^{A^n}$ is an $(\mathcal{A}, \mathcal{B})$-clonoid if for all $n, k \in \mathbb{N}$

- $f_1, \dots, f_n \in \mathcal{C}^{(k)}, g \in \mathcal{B}^{(n)} \Rightarrow g \circ (f_1, \dots, f_n) \in \mathcal{C}^{(k)}$,
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$\mathcal{C} \subseteq \mathcal{O}_{(A,B)}$, the n-ary part of $\mathcal{C}$ is given by $\mathcal{C}^{(n)} := \mathcal{C} \cap B^{A^n}$.

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- Kawałek, Kompatscher, Krzaczkowski (2024) Circuit equivalence in 2-nilpotent algebras.

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Theorem (Sparks 2019)

Let $\mathcal{C}_{\mathbf{AB}}$ *denote the set of all clonoids with finite source* A ($|A| > 1$) *and target algebra* $\mathbf{B}$ *of size* 2. *Then*

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- Lehtonen (2026) characterization of cardinalities of all Boolean clonoid lattices.

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1. $\mathcal{C}_{\mathbf{AB}}$ *is finite if* $\mathbf{B}$ *has an NU-term;*
2. $\mathcal{C}_{\mathbf{AB}}$ *has a Cube term but no majority term no info;*
3. $\mathcal{C}_{\mathbf{AB}}$ *has size continuum only if* $\mathbf{B}$ *has neither an NU-term nor a Cube term.*

Uniform generation

Definition

Let $\mathcal{A}$ be a clone. For $k, l, n \in \mathbb{N}$, let $R_n^{l,k}(\mathcal{A})$ be the set of all the operations $r \in (\mathcal{A}^{(l)})^k$ s. t. there are $v \in (\mathcal{A}^{(n)})^k$ and $w \in (\mathcal{A}^{(l)})^n$ with

$$r(\mathbf{x}) = \begin{bmatrix} r_1(\mathbf{x}) \\ \vdots \\ r_k(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} v_1(w_1(\mathbf{x}), \dots, w_n(\mathbf{x})) \\ \vdots \\ v_k(w_1(\mathbf{x}), \dots, w_n(\mathbf{x})) \end{bmatrix} = v \circ w(\mathbf{x}).$$

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We further define $R_n(\mathcal{A}) = \bigcup_{l,k} R_n^{l,k}(\mathcal{A})$. For an algebra $\mathbf{A}$, we will also use the notation $R_n^{l,k}(\mathbf{A}) = R_n^{l,k}(\mathrm{Clo}(\mathbf{A}))$ and $R_n(\mathbf{A}) = R_n(\mathrm{Clo}(\mathbf{A}))$.

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Lemma

Let $\mathcal{A}, \mathcal{B}$ *be clones,* $f \colon A^k \to B$, $g \colon A^l \to B$, *and* $\mathcal{C} = \langle f \rangle_{\mathcal{A},\mathcal{B}}$. *Then* $g \in \langle \mathcal{C}^{(n)} \rangle_{\mathcal{A},\mathcal{B}}$ *if and only if there are* $m \in \mathbb{N}$, $s \in \mathcal{B}^{(m)}$, *and* $r_1 \ldots, r_m \in R_n^{l,k}(\mathcal{A})$ *such that*

$$g = s \circ (f \circ r_1, \ldots, f \circ r_m). \tag{1}$$

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Definition

Let $\mathcal{A}$, $\mathcal{B}$ be clones. We say that $U \subseteq B^{A^k}$ is uniformly generated by its n-ary $(\mathcal{A}, \mathcal{B})$-minors if there are $s \in \mathcal{B}^{(m)}$ and $r_1 \dots, r_m \in R_n^{k,k}(\mathcal{A})$ such that for every $f \in U$:

$$f = s \circ (f \circ r_1, \dots, f \circ r_m). \tag{2}$$

For $U \subseteq \mathcal{O}_{A,B}$, we say that U is uniformly generated by its n-ary $(\mathcal{A}, \mathcal{B})$-minors if $U^{(k)}$ is uniformly generated by its n-ary $(\mathcal{A}, \mathcal{B})$-minors, for every $k \in \mathbb{N}$.

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Theorem

Let $\mathcal{A}$, $\mathcal{B}$ be clones, $n \geq 1$, and let $\mathcal{A}_{\mathrm{const}} = \mathrm{Clo}(\mathcal{A} \cup \{a\}_{a \in A})$ be the clone generated by $\mathcal{A}$ and all constant operations over its set. Let $\mathcal{C}$ be a clonoid from $\mathcal{A}_{\mathrm{const}}$ to $\mathcal{B}$. Then TFAE:

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1. $\mathcal{C}^{(n+1)}$ *is $(n, \mathcal{A}, \mathcal{B})$-UG.*
2. $\mathcal{C}^{(k)}$ *is $(n, \mathcal{A}, \mathcal{B})$-UG for some $k > n$.*

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Definition
For every matrix $X \in \mathbf{F}^{(k+1)\times k}$ with $\mathrm{rk}(X) = k$ and for every vector $\mathbf{a} \in \mathbf{F}^{k+1} \setminus \mathcal{C}(X)$, we define $\theta(X, \mathbf{a}) = \{X + \mathbf{a}\mathbf{u}^T \mid \mathbf{u} \in \mathbf{F}^k\}$. We call every set of the form $\theta(X, \mathbf{a})$ a θ-space.

θ-spaces

Lemma

Let $X_1, X_2 \in \mathbf{F}^{(k+1)\times k}$, *and let* $\mathbf{a}_1, \mathbf{a}_2 \in \mathbf{F}^{k+1}$ *with* $\mathbf{a}_1 \notin \mathcal{C}(X_1)$ *and* $\mathbf{a}_2 \notin \mathcal{C}(X_2)$.

1. *Either* $|\theta(X_1, \mathbf{a}_1) \cap \theta(X_2, \mathbf{a}_2)| \leq 1$ *or* $\theta(X_1, \mathbf{a}_1) = \theta(X_2, \mathbf{a}_2)$.
2. *For* $X_1 \neq X_2$, *there is at most one* θ*-space containing both* X_1 *and* X_2.
3. *All elements of a* θ*-space have rank* k.

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2. *For $X_1 \neq X_2$, there is at most one θ-space containing both X_1 and X_2.*
3. *All elements of a θ-space have rank k.*

Definition

Let $n \leq k$, and let $A = |\mathbf{a}_1, \ldots, \mathbf{a}_n| \in \mathbf{F}^{(k+1)\times n}$, $U = |\mathbf{u}_1^T, \ldots, \mathbf{u}_n^T| \in \mathbf{F}^{n\times k}$ be matrices of rank n. Then, for $0 \leq j \leq n$, let us set $X_j := X_0 + \sum_{i=1}^{j} \mathbf{a}_i \mathbf{u}_i^T$, We call the rank factorization $AU = X_n - X_0$ (n-)admissible if

1. $\mathbf{a}_{i+1} \notin \mathcal{C}(X_i)$ for all $0 \leq i \leq n-1$;
2. $\mathbf{a}_{i+2} \in \mathcal{C}(X_0) \cap \mathcal{C}(X_1) \cap \cdots \cap \mathcal{C}(X_i)$, for all $0 \leq i \leq n-2$.

θ-spaces

We say that $X \in \mathbf{F}^{(k+1)\times k}$ is of type n if $X = X_0 + AU$ for some n-admissible product AU, and we call $X = X_0 + AU$ an (n-)admissible representation of X. We write $\mathfrak{T}_n$ for the set of matrices of type n, and $\mathfrak{T} = \mathfrak{T}_0 \cup \mathfrak{T}_1 \cup \ldots \cup \mathfrak{T}_k$.

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Lemma

Let $X \in \mathfrak{T}_n$ with representation $X = X_0 + AU$, let $\mathbf{u}_1^T, \ldots, \mathbf{u}_n^T$ be the rows of U, and let $\theta(X, \mathbf{a}) \in \Theta_n$. Then

1. $X + \mathbf{a}\mathbf{u}^T \in \mathfrak{T}_{n+1}$ *if* $\mathbf{u} \notin \langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle$;
2. $X + \mathbf{a}\mathbf{u}^T \in \mathfrak{T}_n$ *if* $\mathbf{u} \in \langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle$.

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Lemma

Let $q = |\mathbf{F}|$ *and let* $X \in \mathfrak{T}_n$, *for* $1 \leq n \leq k$. *Then*

1. X *is in exactly one* θ*-space of type* $n-1$*;*
2. X *is in exactly* q^{k-n} θ*-spaces of type* n*;*
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Theorem

Let $k \in \mathbb{N}$, let $\mathbf{F}$ be a finite field of order q, and let $\mathbf{R}$ be a regular module in which q is invertible. Let $X_0 \in \mathbf{F}^{(k+1)\times k}$ be of rank $\mathrm{rk}(X_0) = k$. *Then*

$$\delta_{X_0}(X) = \sum_{n=0}^{k} (-1)^n q^{\binom{n}{2}-(1+n)k} \sum_{V \in \Theta_n} \delta_V(X).$$

Thus, δ_{X_0} is in the $(\mathbf{F}^k, \mathbf{R})$-clonoid generated by δ_{Id_k}. This further implies that δ_{X_0} is generated by its k-ary $(\mathbf{F}^k, \mathbf{B})$-minors.

Main results

Theorem

Let $\mathbf{F}$ be a finite field, $\mathrm{k} \in \mathbb{N}$, and let $\mathbf{B}$ be an $\mathbf{R}$-module, such that $|\mathrm{F}|$ is invertible in $\mathbf{R}$. Then $\mathcal{O}_{\mathbf{F}^{\mathrm{k}},\mathbf{B}}$ is uniformly generated by k-ary $(\mathbf{F}^{\mathrm{k}}, \mathbf{B})$-minors.

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Corollary

Let $\mathbf{F}^{\mathrm{k}}$ *be the* k*-dimensional vector space over a finite field* $\mathbf{F}$*, and let* $\mathbf{B}$ *be a* $\mathbf{R}$*-module such that* $|\mathrm{F}|$ *is invertible in* $\mathbf{R}$*. Then every clonoid* $\mathcal{C}$ *from* $\mathbf{F}^{\mathrm{k}}$ *to* $\mathbf{B}$ *is generated by its* k*-ary part, i.e.,* $\mathcal{C} = \langle \mathcal{C}^{(\mathrm{k})} \rangle_{\mathbf{F}^{\mathrm{k}},\mathbf{B}}$.

Main results

Corollary

Let $\mathbf{A} = \mathbf{F}_1^{k_1} \times \cdots \times \mathbf{F}_n^{k_n} \times \mathbf{D}$ *considered as an* $\mathbf{F}_1 \times \cdots \times \mathbf{F}_n \times \mathbf{R}$*-module, where* $\mathbf{D}$ *is a finite distributive* $\mathbf{R}$*-module. Let* l *be the nilpotence degree of the Jacobson radical of* $\mathbf{R}$*, and let* $k = \max(k_1, \ldots, k_n, l)$*. Further, let* $\mathbf{B}$ *be an* $\mathbf{S}$*-module, such that* $|A|$ *is invertible in* $\mathbf{S}$*. Then*

1. *every clonoid from* $\mathbf{A}$ *to* $\mathbf{B}$ *is generated by its* k*-ary part.*
2. *If* $\mathbf{B}$ *is finite, then there are only finitely many clonoids from* $\mathbf{A}$ *to* $\mathbf{B}$*.*

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- By $\mathbf{R}[\mathrm{GL}_i(\mathbf{F})]$ we denote the group ring of $\mathrm{GL}_i(\mathbf{F})$ over $\mathbf{R}$.
- We define $\mathbf{M}_{0,k}(\mathbf{F},\mathbf{B}) = \mathbf{B}$ (as $\mathbf{R}$-module) and, for every $1 \leq i \leq k \in \mathbb{N}$, $\mathbf{M}_{i,k}(\mathbf{F},\mathbf{B})$ the $\mathbf{R}[\mathrm{GL}_i(\mathbf{F})]$-module of all functions $f\colon \mathbf{F}^{i\times k} \to \mathbf{B}$ with $\mathrm{supp}(f) \subseteq \{X \in \mathbf{F}^{i\times k} \mid \mathrm{rk}(X) = i\}$.

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Theorem

Let $\mathbf{S}_i$ be the lattice of $\mathbf{R}[\mathrm{GL}_i(\mathbf{F})]$-submodules of $\mathbf{M}_{i,k}$. Then, the lattice of clonoids from $\mathbf{F}^k$ to $\mathbf{B}$ is isomorphic to $\prod_{i=0}^{k} \mathbf{S}_i$.

Sharp bound

Theorem

Let $\mathbf{A}$ *be a finite* $\mathbf{R}_A$*-module and* $\mathbf{B}$ *be a finite* $\mathbf{R}_B$*-module. Assume that the full clonoid* $\mathcal{O}_{\mathbf{A},\mathbf{B}}$ *is generated by the* m*-ary functions, for some* m*. Then* $m \geq \frac{\log |A|}{\log |R_A|}$*.*

Future work

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- Deal with uniform generation when the source is $\mathbb{Z}_{p^2}^2$;
- Combine the obtained results and Mayr-Wynne '24 to prove that clonoids between finite modules are finitely many iff the modules are coprime;
- Use the results to prove the tractability of subpower membership (SMP) over finite modules (2-nilpotent Mal'cev case);

Future work

- Deal with uniform generation when the source is $\mathbb{Z}_{p^2}^2$;
- Combine the obtained results and Mayr-Wynne '24 to prove that clonoids between finite modules are finitely many iff the modules are coprime;
- Use the results to prove the tractability of subpower membership (SMP) over finite modules (2-nilpotent Mal'cev case);

Thank you for your attention