Dual presentations of frames

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The resulting frame $\langle G \mid R \rangle$ is the initial frame satisfying these conditions.

For example, $\langle a, b \mid a \vee b = 1, a \wedge b = 0 \rangle$ describes the initial frame with two complementary elements. This is the frame of opens of the two-point discrete space 2.

Frame presentations and geometric logic

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This leads to the presentation

$$\begin{aligned} \mathcal{O}X = \big\langle [=x] \text{ for } x \in X \mid \\ & [=x] \wedge [=y] \leq {!}^*[\![x = y]\!], \\ & 1 \leq \bigvee_{x \in X} [=x] \big\rangle \end{aligned}$$

Presentations as equalisers

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$$F(R) \rightrightarrows F(G) \twoheadrightarrow \langle G \mid R \rangle.$$

In the category of locales, we can express this as an equaliser or inserter

$$\bullet \hookrightarrow \mathbb{S}^G \rightrightarrows \mathbb{S}^R,$$

where $\mathbb{S}$ is Sierpiński space and corresponds to the free frame on one generator.

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This allows us to generalise from sets of generators and relations to *locales* of generators and relations, as long as the exponentials $\mathbb{S}^G$ and $\mathbb{S}^R$ exist. We call these generalised presentations.

Dual presentations

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The relations are specified by maps $\mathbb{S}^G \rightrightarrows \mathbb{S}^R$, or equivalently maps $R \times \mathbb{S}^G \rightrightarrows \mathbb{S}$ (or even $R \rightrightarrows \mathbb{S}^{\mathbb{S}^G}$). We could think of these as specifying opens of $\mathbb{S}^G$ for each point of R. In fact, for dual presentations it's better to think of them as specifying *closeds*.

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As with ordinary presentations, we will want to build up these maps from logical formulae.

Building dual presentations

The locale $\mathbb{S}^G$ can be viewed as the space of closed sublocales of G.

Each point $g \in G$ gives a basic closed $[g \in F]$ of $\mathbb{S}^G$ via evaluation. Simple formulae of the form $[f(r) \in F]$ for $r \in R$ then correspond to maps $R \times \mathbb{S}^G \xrightarrow{f \times \mathbb{S}^G} G \times \mathbb{S}^G \xrightarrow{\mathrm{ev}} \mathbb{S}$ (or maps $R \xrightarrow{f} G \hookrightarrow \mathbb{S}^{\mathbb{S}^G}$).

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Using the internal lattice operations on $\mathbb{S}$ or the order-enrichment of **Loc**, we can now take finite meets and joins of these and thus build up maps corresponding to more complex formulae.

(Do note that using closeds instead of opens means that joins of the formulae correspond to *meets* in the order-enrichment and vice versa.)

An example

The theory of (closed) equivalence relations on a compact Hausdorff locale X has basic propositions $[x \sim y]$ for $x, y \in X$ and axioms

- $\vdash [x \sim x]$ for $x \in X$,
- $[x \sim y] \vdash [y \sim x]$ for $x, y \in X$,
- $[x \sim y] \wedge [y \sim z] \vdash [x \sim z]$ for $x, y, z \in X$.

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This gives a presentation with $G = X^2$, $R = X \sqcup X^2 \sqcup X^3$ and maps

$$X \xrightarrow{!} 1 \xrightarrow{\perp} \mathbb{S}^{\mathbb{S}^{X^2}} \qquad X \xrightarrow{\Delta} X^2 \hookrightarrow \mathbb{S}^{\mathbb{S}^{X^2}}$$

$$X^2 \hookrightarrow \mathbb{S}^{\mathbb{S}^{X^2}} \qquad X^2 \xrightarrow{(\pi_2,\pi_1)} X^2 \hookrightarrow \mathbb{S}^{\mathbb{S}^{X^2}}$$

$$X^3 \xrightarrow{\pi_{1,2} \vee \pi_{2,3}} X^2 \hookrightarrow \mathbb{S}^{\mathbb{S}^{X^2}} \qquad X^3 \xrightarrow{\pi_{1,3}} X^2 \hookrightarrow \mathbb{S}^{\mathbb{S}^{X^2}} .$$

Joins indexed by compact locales

In ordinary presentations we can also take arbitrary joins — that is, joins indexed by sets. For dual presentations we can take joins indexed by compact (Hausdorff) locales.

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A join of opens $\bigvee_{i \in I} a_i$ indexed by a set I can be expressed as the image of an open $a \in \mathcal{O}(I \times \mathbb{S}^G)$ under $\pi_2 \colon I \times \mathbb{S}^G \to \mathbb{S}^G$. (This makes sense because the projection π_2 is an open map.)

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A join of closeds indexed by a compact locale I can similarly be defined by taking the image of a closed sublocale of $I \times \mathbb{S}^G$ under the projection $\pi_2 \colon I \times \mathbb{S}^G \to \mathbb{S}^G$. (Indeed, since I is compact, the projection is proper and the image is given by the right adjoint to π_2^*.)

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Given maps $f_1, f_2\colon R \to X$ we can then form a map

$$R \xrightarrow{f} X \times X \xrightarrow{d} \mathbb{S} \xrightarrow{\mathbb{S}^!} \mathbb{S}^G$$

representing $f_1(r) = f_2(r)$ for $r \in R$.

The self-presentation of a compact Hausdorff locale

Just as a discrete locale X has a presentation in terms of X, if X is compact Hausdorff there is a dual presentation of a X in terms of itself.

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The corresponding dual presentation has $G = X$, $R = X^2 \sqcup 1$ and maps

$$X^2 \xrightarrow{\pi_1 \vee \pi_2} X \hookrightarrow \mathbb{S}^{\mathbb{S}^X} \qquad X^2 \xrightarrow{d} \mathbb{S} \xrightarrow{\mathbb{S}^!} \mathbb{S}^{\mathbb{S}^X}$$

$$1 \xrightarrow{\perp} \mathbb{S}^{\mathbb{S}^X} \qquad 1 \xrightarrow{e} \mathbb{S}^{\mathbb{S}^X}$$

where e corresponds to the image of the closed given by $\mathrm{ev}\colon X \times \mathbb{S}^X \to \mathbb{S}$ under $\pi_2\colon X \times \mathbb{S}^X \to \mathbb{S}^X$.

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where e corresponds to the image of the closed given by $\mathrm{ev}\colon X \times \mathbb{S}^X \to \mathbb{S}$ under $\pi_2\colon X \times \mathbb{S}^X \to \mathbb{S}^X$. This indeed presents X.

Another example

Let X be a compact Hausdorff locale. We can use dual presentations to describe a locale $[2^{\mathbb{N}} \twoheadrightarrow X]$ of partial surjections from $2^{\mathbb{N}}$ (Cantor space) to X.

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The basic closed propositions are $[f(s) = x]$ for $s \in 2^{\mathbb{N}}$ and $x \in X$. The axioms are

- $[f(s) = x] \wedge [f(s) = x'] \vdash x = x'$ for $s \in 2^{\mathbb{N}}$ and $x, x' \in X$,
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As before this gives a dual presentation for a locale $[2^{\mathbb{N}} \rightharpoondown\!\!\!\!\rightharpoondown X]$.

The Alexandroff–Hausdorff theorem

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We can use the locale $[2^{\mathbb{N}} \twoheadrightarrow X]$ to remove countability restrictions.

Theorem

Let X be a compact Hausdorff locale. The unique morphism $! \colon [2^{\mathbb{N}} \twoheadrightarrow X] \to 1$ is an effective descent morphism.

Beyond dual presentations

Generalised presentations give a powerful tool for specifying locales.

Many of the ideas we have covered involving dual presentations can actually be applied to generalised presentations more generally.